1. One Solution of
$$x^2y'' - 2y = 0, \qquad (x > 0)$$
on $0 < x < \infty$ is $y_1(x) = x^2$. Find all solutions of
$$x^2y'' - 2y = 2x - 1 \qquad (x > 0)\,.$$

2. One solution of
$$x^2y'' - xy' + y = 0, \qquad (x > 0),$$
is $y_1(x) = x$. Find the solution ψ of
$$x^2y'' - xy' + y = x^2$$
satisfying $\psi(1) = 1$ and $\psi'(1) = 0$.

3. Show that a linear differential equation
$$p_0(x)y'' + p_1(x)y' + p_2(x)y = p_3(x)\lambda y, \qquad (p_0(x) > 0),$$
can be transformed to SL differential equation by multiplying by
$$\frac{1}{p_0(x)} \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right].$$
Using this to transform Chebyshev II equation
$$\left(1 - x^2\right) y'' - 3xy' + n(n+2)y = 0, \qquad (x \in [-1, 1])\,,$$
to SL differential equation.

4. Consider an eigenvalue problem
$$\begin{aligned} -\frac{d^2}{dx^2}y &= \lambda y, \qquad x \in [0, 1] \\ y(0) &= 0, \\ y'(1) &= y(1). \end{aligned}$$
   (a) Is this a Sturm-Liouville system? Is it regular, periodic or singular?
   (b) Find eigenvalues and eigenfunctions.
   (c) Verify that eigenfunctions constistute an orthogonal set. Normalize eigenfunctions suitably.

5. Consider an eigenvalue problem
$$\begin{aligned} -\frac{d^2}{dx^2}y &= \lambda y, \qquad x \in [0, b], \\ y(0) &= 0, \\ y'(0) &= y(b). \end{aligned}$$
   (a) Is this a Sturm-Liouville system?
   (b) Find all eigenvalues and eigenfunctions.
   (c) Are eigenfunctions are orthogonal?

6. Show that the eigenvalues and eigenfunctions of the SL problem

$$\begin{aligned} \frac{d}{dx}\left[(1+x)^2 \frac{dy}{dx}\right] + \lambda y &= 0, \qquad x \in [0,1], \\ y(0) = y(1) &= 0 \end{aligned}$$

are given by

$$\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 + \frac{1}{4}$$

and

$$y_n = \frac{1}{\sqrt{1+x}} \sin\left(n\pi \frac{\ln(1+x)}{\ln 2}\right).$$

Plot the first few functions and verify the theorem regarding the zeros of the eigenfunctions.