1 Linear Equations with Regular Singular Points

Consider the equation

$$x^2y'' + a(x)xy' + b(x)y = 0$$

where a and b are analytic with convergent power series expansions (about $x = 0$) for

$$|x| < r_0, \qquad r_0 > 0.$$

Clearly $x = 0$ is a regular singular point. The indicial equation is given by

$$r(r-1) + a(0)r + b(0) = 0.$$

Let r_1 and r_2 be the two solutions of the indial equation such that $\text{Re}r_1 \geq \text{Re}r_2$.

1. $r_1 \neq r_2$ and $r_1 - r_2$ is not a positive integer:

   Two solutions exist with the form

   $$y_1 = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k$$

   and

   $$y_2 = |x|^{r_2} \sum_{k=0}^{\infty} d_k x^k$$

   with both series convergent for $|x| < r_0$. The coefficients c_k and d_k can be uniquely determined by substituting solutions in the differential equation.

2. $r_1 = r_2$:

   Two solutions exist with the form

   $$y_1 = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k$$

   and

   $$y_2 = |x|^{r_1+1} \sum_{k=0}^{\infty} d_k x^k + (\log|x|)\, y_1$$

   with both series convergent for $|x| < r_0$. The coefficients c_k and d_k can be uniquely determined by substituting solutions in the differential equation.

3. $r_1 - r_2$ is a positive integer:

   Two solutions exist with the form

   $$y_1 = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k$$

   and

   $$y_2 = |x|^{r_2} \sum_{k=0}^{\infty} d_k x^k + c(\log|x|)\, y_1$$

   with both series convergent for $|x| < r_0$. The coefficients c_k, d_k and c can be uniquely determined by substituting solutions in the differential equation. It may happen that c is zero.

1.1 Example of case 1

Consider the differential equation

$$x^2y'' + \left(\frac{5}{3}\right)xy' + xy = 0.$$

Here $a(x) = 5/3$ and $b(x) = x$ both analytic for all x. The indicial equation

$$r(r-1) + \frac{5}{3}r + 0 = 0$$

has solutions $r_1 = 0$ and $r_2 = -2/3$. This is case 1 above.

Then, let

$$y_1 = \sum_{k=0}^{\infty} c_k x^k$$

and substitute in differential equation

$$\sum_{k=0}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} \frac{5}{3}kc_k x^k + \sum_{k=0}^{\infty} c_k x^{k+1} = 0.$$

Clearly, coefficients of x^k must be zero: Thus for $k \geq 1$

$$\begin{aligned} k(k+\frac{2}{3})c_k + c_{k-1} &= 0 \\ \therefore \quad c_k &= -\frac{3}{k(3k+2)}c_{k-1} \\ &= \frac{(-1)^k 3^k}{k!5\cdot 8\cdots(3k+2)}c_0 \end{aligned}$$

Thus the first solution is

$$y_1 = c_0\left[1 + \sum_{k=1} \frac{(-1)^k 3^k}{k!5\cdot 8\cdots(3k+2)}x^k\right]$$

The second solution is given by

$$y_1 = x^{-2/3}\sum_{k=0}^{\infty} c_k x^k.$$

The recurrence relation is

$$\begin{aligned} c_k &= -\frac{3}{k(3k-2)}c_{k-1} \\ &= \frac{(-1)^k 3^k}{k!1\cdot 4\cdots(3k-2)}c_0 \end{aligned}$$

The second solution is

$$y_2 = x^{-2/3}c_0\left[1 + \sum_{k=1} \frac{(-1)^k 3^k}{k!1\cdot 4\cdots(3k-2)}x^k\right].$$

Both series are convergent:

$$\lim_{k\to\infty}\left|\frac{c_k x^k}{c_{k-1}x^{k-1}}\right| = \lim_{k\to\infty}\left|\frac{-3x}{k(3k\pm 2)}\right| = 0$$

It is easy to check that the two solutions are linearly independent.

1.2 Example of Case 2

Consider

$$x^2y'' + xy' + x^2y = 0$$

Here $a(x) = 1$ and $b(x) = x^2$ and both are analytic for all x. Thus indicial equation is

$$r(r-1) + r + 0 = 0.$$

The only solution is $r = 0$. This is same as case 2 described above.

Then, let

$$y_1 = \sum_{k=0}^{\infty} c_k x^k$$

and substitute in differential equation

$$\sum_{k=0}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} kc_k x^k + \sum_{k=0}^{\infty} c_k x^{k+2} = 0.$$

Clearly, coefficients of x^k must be zero: Then, coefficient of x

$$c_1 = 0$$

Thus for $k \geq 2$

$$\begin{aligned} k^2c_k + c_{k-2} &= 0 \\ \therefore \quad c_k &= -\frac{1}{k^2}c_{k-2} \\ &= \frac{(-1)^{k/2}}{2^k\left(\frac{k}{2}!\right)^2}c_0 \end{aligned}$$

Thus for odd k, clearly $c_k = c_{k-2} = \cdots = c_3 = c_1 = 0$.

For even k,

$$c_k = \frac{(-1)^{k/2}}{2^k\left(\frac{k}{2}!\right)^2}c_0$$

Thus the first solution is

$$\begin{aligned} y_1 &= c_0 \sum_{k=0,2,\ldots} \frac{(-1)^{k/2}}{2^k\left(\frac{k}{2}!\right)^2}x^k \\ &= c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}x^{2m} \end{aligned}$$

The second solution is given by

$$\begin{aligned} y_2 &= x\sum_{k=0}^{\infty} \widetilde{b}_k x^k + \log(x)y_1 \\ &= \sum_{k=0}^{\infty} b_k x^k + \log(x)y_1 \end{aligned}$$

such that $b_0 = 0$. Putting y_2 in the differential equation, we get

$$b_1x + 2^2b_2x^2 + \sum_{k=3}^{\infty}(k^2b_k + b_{k-2}) + 2xy_1' = 0$$

Thus

$$b_1x + 2^2b_2x^2 + \sum_{k=3}^{\infty}(k^2b_k + b_{k-2}) = -2c_0\sum_{m=0}^{\infty}\frac{(-1)^m 2m}{2^{2m}(m!)^2}x^{2m}$$

Comparing the coefficients of x^k from both sides,

$$\begin{aligned} b_1 &= b_3 = \cdots = 0 \\ b_2 &= c_0/4 \end{aligned}$$

and for $k \geq 4$ or $m \geq 2$,

$$(2m)^2b_{2m} + b_{2m-2} = \frac{(-1)^{m+1}m}{2^{2m-2}(m!)^2}c_0$$

with $b_0 = 0$ and $b_2 = c_0/4$. Solving this recurrence relation, we get

$$b_{2m} = \frac{(-1)^{m+1}}{2^{2m-2}(m!)^2}\left[1 + \frac{1}{2} + \cdots + \frac{1}{m}\right]c_0.$$

Thus the second solution is given by

$$y_2 = c_0\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^{2m-2}(m!)^2}\left[1 + \frac{1}{2} + \cdots + \frac{1}{m}\right]x^{2m} + \log(x)y_1.$$

1.3 Example of case 3: Bessel Equation

The Bessel equation is given by

$$x^2y'' + xy' + \left(x^2 - \alpha^2\right)y = 0.$$

Clearly the previous section, we considered a special case $\alpha = 0$. Thus let $\alpha > 0$. The indicial equation is

$$r^2 - \alpha^2 = 0 \implies r = \pm\alpha.$$

If α is an integer or half odd integer, we have case 3. Assume that $\alpha = n$ some integer.

The first solution is

$$y_1 = x^n\sum_{k=0}^{\infty}c_kx^k$$

Then, substitution of y_1 in the differential equation yields,

$$0 \cdot c_0x^n + \left[(n+1)^2 - n^2\right]c_1x^{n+1} + \sum_{k=2}^{\infty}\left\{\left[(n+k)^2 - n^2\right]c_k + c_{k-2}\right\}x^{n+k} = 0$$

Then, comparing coefficients, we get $c_1 = 0$ and for $k \geq 2$,

$$[k(2n+k)]\,c_k + c_{k-2} = 0$$

Consequently, $c_k = 0$ for all odd k. And for even k

$$\begin{aligned} c_k &= -\frac{1}{k(2n+k)}c_{k-2} \\ &= \frac{(-1)^{k/2}}{2^{k/2}\left(\frac{k}{2}\right)!(2n+2)(2n+4)\cdots(2n+k)}c_0 \\ &= \frac{(-1)^{k/2}n!}{2^k\left(\frac{k}{2}\right)!\left(n+\frac{k}{2}\right)!}c_0 \end{aligned}$$

Or put $k = 2m$, then

$$c_m = \frac{(-1)^m n!}{2^{2m}\,(m)!\,(n+m)!} c_0$$

usually, c_0 is chosen to be $(2^n n!)^{-1}$, then the first solution can be written as

$$y_1 = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{(m)!\,(n+m)!} \left(\frac{x}{2}\right)^{2m}. \tag{1}$$

This solution is called the Bessel Function of the first kind and denoted by $J_n(x)$.

The second solution is given by

$$y_2 = x^{-n} \sum_{k=0}^{\infty} d_k x^k + c\log(x) y_1$$

Substituting in DE, we get

$$0 \cdot d_0 x^{-n} + \left[(-n+1)^2 - n^2\right] d_1 x^{-n+1} + \sum_{k=2}^{\infty} \left\{\left[(-n+k)^2 - n^2\right] d_k + d_{k-2}\right\} x^{-n+k} + 2cxy_1' = 0$$

Substitute y_1' from equation 1,

$$\left[(-n+1)^2 - n^2\right] d_1 x^{-n+1} + \sum_{k=2}^{\infty} \left\{\left[(-n+k)^2 - n^2\right] d_k + d_{k-2}\right\} x^{-n+k} = -2c \sum_{m=0}^{\infty} c_m (2m+n) x^{2m+n}$$

Multiply by x^n,

$$\left[(-n+1)^2 - n^2\right] d_1 x + \sum_{k=2}^{\infty} \left\{\left[(-n+k)^2 - n^2\right] d_k + d_{k-2}\right\} x^k = -2c \sum_{m=0}^{\infty} c_m (2m+n) x^{2m+2n}$$

Upon comparing coefficients of x^k for *odd* k, we get (rhs has no terms in odd powers of x)

$$\begin{aligned} d_1 &= 0 \\ d_k &= -\frac{1}{k(k-2n)} d_{k-2} \qquad \text{odd } k \end{aligned}$$

The denominator will never become zero, since k is odd and $2n$ is even. Thus, $d_1 = d_3 = \cdots = 0$.

Now compare the coefficients of x^k for even k. Note rhs the smallest power of x is $2n$. Thus

$$\begin{aligned} d_k &= -\frac{1}{k(k-2n)} d_{k-2} \qquad \text{even } k < 2n-2 \\ d_{2n-2} &= -2cc_0 n \qquad \text{even } k = 2n \\ d_k &= -\frac{1}{k(k-2n)} \left[-2cc_m(2m+n) - d_{k-2}\right] \qquad \text{even } 2m+2n = k > 2n \end{aligned}$$

First solve first part, that is $k < 2n-2$

$$d_k = \frac{(-1)^{k/2}}{2^k\,(k/2)!\,(-n+1)\,(-n+2)\cdots(-n+k/2)} d_0$$

When

$$d_{2n-2} = -2nc\frac{1}{2^n n!} = \frac{c}{2^{n-1}(n-1)!}$$

Now, d_{2n} is undetermined. Choose $d_{2n} = -\frac{cc_0}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$. Then the final solution becomes

$$\begin{aligned} y_2 &= d_0 x^{-n} + d_0 x^{-n} \sum_{j=0}^{n-1} \frac{x^{2j}}{2^{2j} j!(n-1)\cdots(n-j)} - \frac{cc_0}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &\quad - \frac{c}{2} \sum_{m=1}^{\infty} c_{2m} \left(\sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{m+n} \frac{1}{l}\right) x^{n+2m} + c(\log x) y_1. \end{aligned}$$