1. Prove the mean value theorem: Let P be an interior point of a volume$V$. Let y be a solution of the Laplace equation in V. Then $y(P)$ is the average of y over the surface of any sphere in V centered about P. [Hint: Use the integral equation.] Prove that the solution of the Laplace equation cannot have a maximum or a minimum in V.

2. Consider the Laplace equation $\nabla^2\phi = 0$ in a volume V with boundary S.

   (a) Prove using the Green's identity, that for a function f,

   $$\int_V \left(f\nabla^2 f + |\nabla f|^2\right) dv = \oint_S f\left(\nabla f \cdot \hat{\mathbf{n}}\right) dS.$$

   (b) Prove that the solution (assuming that it exists) to the Laplace equation in V with either Dirichlet or Neumann boundary conditions must be unique.

3. Prove that the Green's function for Laplace equation must be symmetric under exchange of its arguments, that is, $G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r})$. [Note: This result is true for all self-adjoint operators. Try and prove this result for Green's functions of Sturm-Liouville equations.]

4. Show that the Dirichlet Green's function for the unbounded space between $z = 0$ and $z = L$ planes is given by

   $$G(\mathbf{r}, \mathbf{r}') = \frac{4}{L}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) I_m\left(\frac{n\pi}{L}\rho_<\right) K_m\left(\frac{n\pi}{L}\rho_>\right).$$

5. For the same geometry of problem 4, show that alternate form of the Green's function is

   $$G(\mathbf{r}, \mathbf{r}') = 2\sum_{m=-\infty}^{\infty}\int_0^k dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') \frac{\sinh(kz_<)\sinh\left[k(L - z_>)\right]}{\sinh(kL)}.$$

6. Consider two parallel conducting plates $z = 0$ and $z = L$. The potential on the $z = 0$ plate is zero, and on $z = L$ is given by

   $$\Phi(\rho, \phi, L) = \begin{cases} V & \rho \leq a \\ 0 & \rho > a. \end{cases}$$

   (a) Show that the potential between the plates cane be written as

   $$\Phi(\rho, \phi, z) = V\int_0^{\infty} d\lambda J_1(\lambda) J_0(\lambda\rho/a)\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}$$

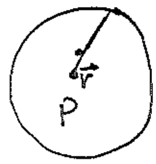
Tutorial 7

Q1. (a) Given, for a volume V,

$$\phi(\vec{r}) = \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \phi}{\partial n'} - \phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \right] ds'$$

if ϕ is a solⁿ of the Laplace Eq.

Let S be a spherical shell of radius a about $\vec{r}$. Then $\frac{1}{R} = \frac{1}{|\vec{r}-\vec{r}'|}$



$\therefore \quad \frac{1}{R} = \frac{1}{a}$ on S and $\hat{n} = \widehat{(\vec{r}-\vec{r}')}$

$$\hat{n} \cdot \nabla' \frac{1}{|\vec{r}-\vec{r}'|} = -\widehat{(\vec{r}-\vec{r}')} \cdot \frac{(\vec{r}-\vec{r}')}{a^3} = -\frac{1}{a^2}$$

and

$$\oint_S \nabla'\phi \cdot \hat{n}' ds = \int_V \nabla^2 \phi \, ds = 0 .$$

$$\therefore \quad \phi(\vec{r}) = \frac{1}{4\pi a^2} \oint_S \phi \, ds' .$$

(b) If ϕ has a maximum at p then for some ϵ-nbd of p $\quad \phi(\vec{r}) \leqslant \phi(p)$. Thus on ϵ-ball

$$\frac{1}{4\pi a^2} \int \phi \, ds < \phi(p)$$

Strictly less than

This is in constrast to previous Result.

Q2. (a) Now by divergence theorem $\int_V \nabla \cdot A \, dv = \oint_S A \cdot n \, ds$

Choose $A = f \nabla f$, $\quad \nabla \cdot A = \nabla \cdot (f \nabla f) = f \nabla^2 f + \nabla f \cdot \nabla f$

$$= f \nabla^2 f + |\nabla f|^2$$

$$\therefore \quad \int_V (f \nabla^2 f + |\nabla f|^2) dv = \oint_S f \frac{\partial f}{\partial n} ds \quad \text{—— (1)}$$

(b) if ϕ_1 is solⁿ of $\nabla^2 \phi_1 = 0$ with DBC $\quad \phi_1(\vec{r}) = \eta(\vec{r})$ on S

So if ϕ_2, i.e. $\nabla^2 \phi_2 = 0 \quad$ " $\quad \phi_2(\vec{r}) = \eta(\vec{r})$ on S

This implies that $\phi = \phi_1 - \phi_2$ satisfies

$$\nabla^2\phi = 0 \quad \text{s.t.} \quad \phi = 0 \text{ on } S.$$

Thus if $f = \phi$ in ①

$$\int_V (\nabla\phi)^2 dv = 0.$$

$\Rightarrow \quad \nabla\phi = 0$ on V

$\Rightarrow \quad \phi = \text{Const}$ on V. but the const must be zero since $\phi = 0$ on S.

$\Rightarrow \quad \phi_1 = \phi_2$ ~~+ const~~.

Q3. Let r_1, r_2 be two points in V bounded by S. Consider Dirichlet Green's function.

$$\nabla^2 G(\bar{r}, \bar{r}') = -4\pi\delta(r - r') \quad \text{and } G(r, r') = 0 \text{ on } S.$$

Then

$$G(r, r_1)\nabla^2 G(r, r_2) - G(r, r_2)\nabla^2 G(r, r_1)$$
$$= [G(r, r_1)\delta(r - r_2) - G(r, r_2)\delta(r - r_1)](-4\pi)$$

Integrate on Surface.

$$\text{RHS} = \int_V [G\nabla^2 G(r, r_2) - G(r, r_2)\nabla^2 G(r, r_1)]dv$$

$$= \int_V \nabla\cdot[G(r, r_1)\nabla G(r, r_2) - G(r, r_2)\nabla G(r, r_1)]dv$$

$$= \oint_S [G(r, r_1)\nabla G(r, r_2) - G(r_1, r_2)\nabla G(r, r_2)]ds$$

$= 0$ since $G = 0$ on S [clearly will apply to Neumann case too.]

$$\text{RHS} = G(r_2, r_2) - G(r_1, r_2)$$

$$\Rightarrow \quad G(r_1, r_2) = G(r_2, r_1).$$

Q4: Assume that

$$G(\vec{r},\vec{r}') = \frac{4}{L}\sum_{m,n} g_m(r,r') \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right) e^{-im(\phi-\phi')}$$

Then $G=0$ if $z=0$ or $z=L$.

Now, $$\nabla^2 G = \frac{4}{L}\sum_{m,n}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}g_m\right) - \left(\frac{m^2}{r^2}+k^2\right)g\right]e^{im(\phi-\phi')} \times \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)$$

$$= -4\pi\,\delta^3(\vec{r}-\vec{r}') = -\frac{4\pi}{r}\delta(r-r')\,\delta(\phi-\phi')\,\delta(z-z')$$

Thus if $\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \left(\frac{m^2}{r^2}+k^2\right)\right]g = -\frac{1}{r}\delta(r-r')$

then $$\nabla^2 G = -\frac{4\pi}{\pi L}\delta(r-r')\sum_m e^{im(\phi-\phi')}\sum_n \sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)$$

$$= -\frac{4\pi}{\pi L r}\delta(r-r')\cdot 2\pi\,\delta(\phi-\phi')\cdot\frac{L}{2}\,\delta(z-z')$$

$$= -\frac{4\pi}{r}\delta(r-r')\,\delta(\phi-\phi')\,\delta(z-z') = -4\pi\,\delta^3(\vec{r}-\vec{r}').$$

The Eq. for g_{mn} is then $\rho = kr$

$$\cancel{\frac{1}{\rho}}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) - \left(\frac{m^2}{\rho}+\rho\right)g = -\delta(\rho-\rho')$$

$\therefore$ $g_1 = I_m(kr)$, $g_2 = K_m(kr)$

$g_1(0) = 0$, $g_2(\infty) = 0$

from properties of Bessel fns.

$$W(g_1,g_2) = -\frac{1}{kr} = -\frac{1}{\rho}.$$

$$c = -\frac{1}{P(\rho')W(\rho')} = 1$$

$$\therefore\quad g_{m,n}(\rho) = I_m(k_n r_<)\,K_m(k_n r_>)$$

$$\therefore\quad G(\vec{r},\vec{r}') = \frac{4}{L}\sum_{m,n} e^{im(\phi-\phi')}\cdot\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z'}{L}\right)\cdot I_m\left(\frac{n\pi}{L}r_<\right)K_m\left(\frac{n\pi}{L}r_>\right).$$

Q5. Orthogonality of Bessel Fns.

$$\int_0^\infty J_m(kr)\, J_m(k'r)\, r\, dr = \frac{1}{k}\,\delta(k-k')$$

$$\therefore \quad \delta(r-r') = \int_0^\infty A_m(k)\, J_m(kr)\, dk$$

$$\therefore \quad \int r\, J_m(k'r)\, \delta(r-r')\, dr = \int_0^\infty A_m(k) \int_0^\infty r\, J_m(k'r)\, J_m(kr)\, dr\, dk$$

$$r'\, J_m(k'r') = \frac{1}{k'}\, A_m(k')$$

$$\therefore \quad A_m(k') = \cancel{\frac{1}{k}}\,(k'r')\, J_m(k'r')$$

$$\Rightarrow \quad \frac{\delta(r-r')}{r'} = \int_0^\infty k\, J_m(kr)\, J_m(kr')\, dk$$

$$\therefore \quad G(r,r') = 2\sum_{m=-\infty}^{\infty} \int_0^\infty dk\, J_m(kr)\, J_m(kr') \cdot e^{im(\phi-\phi')} \cdot g_{mk}(z, z')$$

$$\therefore \quad \nabla^2 G = 2\sum_m \int dk \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] J_m(kr)\, J_m(kr')\, e^{im(\phi-\phi')}\, g_{m,k}(z, z')$$

$$= 2\sum_m \int dk \left[\left(\frac{m^2}{r^2} - k^2\right) - \frac{1}{r^2}m^2 + \frac{\partial^2}{\partial z^2} \right] J_m(kr)\, J_m(kr')\, e^{im(\phi-\phi')}\, g_{m,k}(z-z')$$

$$\therefore \text{ if } \quad \left(\frac{\partial^2}{\partial z^2} - k^2\right) g_{m,k}(z, z') = -\, k\, \delta(z-z')$$

$$\text{then} \quad \nabla^2 G = -2\sum_m \int dk\, \delta(z-z')\, J_m(kr)\, J_m(kr')\, e^{im(\phi-\phi')}$$

$$= -2 \cdot \delta(z-z') \cdot 2\pi\, \delta(\phi-\phi') \cdot \frac{\delta(r-r')}{r'} = -4\pi\,\delta^3(\bar{r}-\bar{r}')$$

$$g_m(z, z') = \frac{\sinh(kz_<)\, \sinh(k(L-z_>))}{\sinh(kL)}$$

Q5. Do the same as in Q4 Except put $\frac{d^2}{dz^2}Q(z) = +k^2 Q(z)$.

Q6. Using Result of Q5.

$$\therefore \quad \phi(\rho,\phi,z) = \int \frac{\partial G(r,r')}{\partial n}\, \phi(\rho,\phi,\theta)\; r\,dr\,d\theta$$

Since $\displaystyle\int e^{im(\theta-\theta')}\,d\theta = 2\pi$ if $m=0$

$= 0$ otherwise.

$$\therefore \quad \phi(\rho,\phi,z) = \int_0^\infty dk\; V \int_0^a r\,dr\; J_0(kr)\, J_1(kr').$$

$$\hat{n} = +\hat{k} \Rightarrow \frac{\partial G}{\partial n} = \cancel{+}\,(\quad)\; \frac{\cancel{\cosh}(kz')\,\cancel{\cosh}(kL-z')}{\sinh(kL)}$$

$$= -(\quad)\;\frac{\sinh(kz')}{\sinh(kL)}$$

$$\Rightarrow \quad \phi(\rho,\phi\, z) = \int_0^\infty dk\; V\, \frac{\sin kz'}{\sinh(kL)}\, J_0(k\rho')\int r\,dr\; J_0(k\rho)$$

$$= \int_0^\infty \frac{2V \sinh kz'}{\sinh(kL)}\, J_0(kr')\, J_1(ka)$$

$\Rightarrow$ Result.