1. Starting with the generating function for the Bessel functions, show the following:

   (a) $J_n(x) = (-1)^n J_n(-x)$.

   (b) $\exp(iz\cos\theta) = \sum_{-\infty}^{\infty} i^m J_m(z)\, e^{im\theta}$.

   (c) $\cos x = J_0(x) + 2\sum_1^{\infty}(-1)^n J_{2n}(x)$ and $\sin x = 2\sum_0^{\infty}(-1)^n J_{2n+1}(x)$.

2. Starting with the series form for the Bessel functions, show the following:

   (a) $\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$.

   (b) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

   (c) $x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2) J_n(x) = 0$.

3. Show that between any two consecutive zeroes of $J_n(x)$ there is one and only one zero of $J_{n+1}(x)$. To prove this, first prove that

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \quad \text{and} \quad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n-1}(x)$$

   using problem 2(a) and 2(b).

4. Prove $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$using the integral representation of the Bessel functions.

5. A plane wave (wavelength λ) is incident normally on a circular apperture (radius a) as shown in the figure. Show that the amplitude of the wave emitted in the direction that makes an angle α with z axis is given by

$$\Phi \sim \int_0^a r\,dr \int_0^{2\pi} e^{ibr\cos\theta} d\theta$$

   where $b = 2\pi \sin\alpha/\lambda$. Thus, show that

$$\Phi \sim \frac{\lambda a}{\sin\alpha} J_1\left(\frac{2\pi a}{\lambda}\sin\alpha\right).$$

   Plot the intensity.

6. Show the following:

$$\begin{aligned}
\left(a^2 - b^2\right)\int_0^P J_n(ax)\, J_n(bx)\, x\,dx &= P\left[bJ_n(aP)\frac{d}{d(bx)}J_n(bP) - aJ_n(bP)\frac{d}{d(ax)}J_n(aP)\right], \\
\int_0^P \left[J_n(ax)\right]^2 x\,dx &= \frac{P^2}{2}\left\{\left[\frac{d}{d(ax)}J_n(aP)\right]^2 - \left(1 - \frac{n^2}{a^2P^2}\right)\left[J_n(aP)\right]^2\right\}, \\
\int_0^a \left[J_n\left(\rho_{nm}\frac{x}{a}\right)\right]^2 x\,dx &= \frac{a^2}{2}\left[J_{n+1}(\rho_{nm})\right]^2.
\end{aligned}$$

Tutorial-5

Q1. The generating function. $G(x,t) = \exp\left[\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)\right]$

(a) $$\sum_n J_n(x)\, t^n = \exp\left[\left(\frac{x}{2}\right)\left(t-\frac{1}{t}\right)\right]$$

$$= \exp\left[\left(-\frac{x}{2}\right)\left((-t)-\frac{1}{(-t)}\right)\right]$$

$$= \sum_n J_n(-x)(-t)^n = \sum_n J_n(x)(-1)^n t^n$$

$$\Rightarrow \cancel{J_n(x)} \quad \underline{J_n(-x) = (-1)^n J_n(x)}$$

(b) Let $t = i e^{i\theta}$, $\quad t - \frac{1}{t} = i\left[e^{i\theta} + e^{-i\theta}\right] = 2i\cos\theta$.

$$\Rightarrow \quad G(x,t) = \sum_m J_m(x)\, t^m$$

$$\Rightarrow \quad e^{ix\cos\theta} = \sum_m J_m(x)\left(i e^{i\theta}\right)^m$$

$$\underline{e^{ix\cos\theta} = \sum_m i^m J_m(x)\, e^{im\theta}}$$

(c) put $t = i$, then $(t - 1/t) = 2i$

$$e^{ix} = \sum_{n=-\infty}^{\infty} J_n(x) \cdot i^n$$

$$= J_0(x) + \sum_{n=1}^{\infty}\left(J_n(x)\, i^n + J_{-n}(x)\, i^{-n}\right)$$

$$= J_0(x) + \sum_{n=1}^{\infty} 2 i^n J_n(x) \qquad \because J_{-n}(x) = (-1)^n J_n(x).$$

<u>Realpart</u>

$$\cos x = J_0(x) + 2\sum_{n,\,\text{even}} (-1)^{n/2} J_n(x)$$

$$\sin x = 2\sum_{n,\,\text{odd}} (-1)^{(n-1)/2} J_n(x).$$

Q2. Series form for Bessel functions:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

(a)
$$J_{n-1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n-1+s)!} \left(\frac{x}{2}\right)^{n-1+2s}$$

$$= \underbrace{\frac{2n}{x}\,\frac{1}{n!}\left(\frac{x}{2}\right)^{n}}_{s=0 \text{ term}} + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \cdot \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{(n+s)\,2}{x}$$

$$J_{n+1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,(n+1+s)!} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{(s-1)!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s-1} \qquad s \to s-1$$

$$= -\sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s} \cdot \frac{2s}{x}.$$

$$\Rightarrow J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$$

(b) Also

$$J_{n-1} - J_{n+1} = \frac{2n}{n!}\left(\frac{x}{2}\right)^{n-1} + \sum_{s=1}^{\infty} \frac{(-1)^s}{s!\,(n+s)!} \left(\frac{x}{2}\right)^{n+2s-1} 2\,(n+2s)$$

$$= 2J_n'(x).$$

(c) From (a) and (b): $2J_{n-1}(x) = \frac{2n}{x} J_n(x) + J_n'(x)$

$\therefore \quad x J_n' = x J_{n-1} - n J_n.$

$\therefore \quad x J_n'' + J_n' = x J_{n-1}' + J_{n-1} - n J_n'$ differentiate.

$\therefore \quad x J_n'' + (n+1) J_n' - x J_{n-1}' - J_{n-1} = 0$ ~~Every~~ shift to RHS.

$\therefore \quad x^2 J_n'' + (n+1) x J_n' - x^2 J_{n-1}' - J_{n-1} = 0$ multy by x.

$\therefore \quad x^2 J_n'' + x J_n' + \underline{n x J_{n-1}} - n^2 J_n \; \underline{- x^2 J_{n-1}' - J_{n-1}} = 0$

$\therefore \quad x^2 J_n'' + x J_n' - n^2 J_n + x^2\left[\frac{n-1}{x} J_{n-1} - J_{n-1}'\right] = 0$

$\therefore \quad x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0$

Q3. From Q 2(a) and (b)

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

and $$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Adding the two eq.

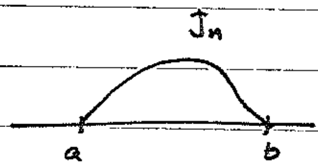
$$\frac{2n}{x} J_n(x) + 2J_n'(x) = 2J_{n-1}(x)$$

$$\Rightarrow \quad n x^{n-1} J_n + x^n J_n' = x^n J_{n-1}$$

$$\Rightarrow \quad [x^n J_n]' = x^n J_{n-1} \quad \text{—— (1)}$$

Similarly $$[x^{-n} J_n]' = -x^{-n} J_{n+1} \quad \text{—— (2)}$$

by integrating (1) from a to b, where a, b are zeroes of J_n,



$$\int_a^b \frac{d}{dx}[x^n J_n]\,dx = \int_a^b x^n J_{n-1}(x)\,dx$$

$$\Rightarrow \int_a^b x^n J_{n-1}(x)\,dx = 0 \Rightarrow J_{n-1}(x) \text{ changes sign in } [a,b]$$

Do the same for second eq: then J_n must change sign between zeroes of J_{n-1}.

$\Rightarrow$ J_{n-1} has one and only one zero between two zeroes of J_n.

Q4. The integral representation of Bessel fn is

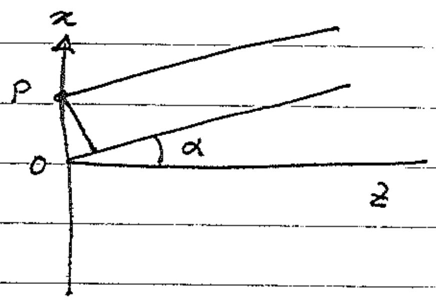
$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos[n\theta - x\sin\theta]\,d\theta$$

Then $$J_n'(x) = \frac{1}{\pi}\int_0^{\pi} -\sin[n\theta - x\sin\theta]\cdot(-\sin\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left\{\cos[(n-1)\theta - x\sin\theta] - \cos[(n+1)\theta - x\sin\theta]\right\}d\theta$$

$$\therefore\ 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x).$$

Q5. Let $P = (x,y) = (r,\theta,0)$



Path diff w.r.t. the ray from O

$$= x\sin\alpha$$

$\therefore$ phase diff

$$= \frac{2\pi x}{\lambda}\sin\alpha = \frac{2\pi}{\lambda} r\cos\theta\sin\alpha$$

$$= br\cos\theta \qquad \text{Let } b = \frac{2\pi\sin\alpha}{\lambda}.$$

Then amplitude in direction of α

$$I_{tot} = I_0\int_0^a\int_0^{2\pi} r\,dr\,d\theta\cdot\exp[ibr\cos\theta]$$

$$= I_0\int_0^a r\,dr\ J_0(br)\cdot 2\pi$$

Remember $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \Rightarrow J_0'(x) = -J_1(x)$

$$\therefore\ I_{tot} = 2\pi I_0\cdot\frac{1}{b^2}\cdot(br)J_1(br)\Big|_0^a \qquad \frac{d}{dx}[xJ_1] = xJ_0.$$

$$\sim \frac{1}{b}J_1(ba)$$

$$\sim \frac{\lambda}{\sin\alpha}\cdot J_1\!\left(\frac{2\pi a}{\lambda}\sin\alpha\right)$$

Intensity: $$I_{tot}^2 \sim \left[\frac{J_1\left(\frac{2\pi a}{\lambda}\sin\alpha\right)}{\sin\alpha.}\right]^2$$

Q6. Now,

$$x^2 J_n''(ax) + x J_n(ax) + (a^2x^2 - n^2) J_n(ax) = 0 \qquad \times J_n(bx)$$
$$x^2 J_n''(bx) + x J_n(bx) + (b^2x^2 - n^2) J_n(bx) = 0 \qquad \times J_n(ax)$$

$\therefore$ Thus

$$x^2 \frac{dW}{dx} + x W + (a^2 - b^2) J_n(ax) J_n(bx) x^2 = 0$$

Here $W = J_n(bx) J_n'(ax) - J_n(ax) J_n'(bx)$

$$\therefore \frac{d}{dx}(xW) = (b^2 - a^2)\, x J_n(ax) J_n(bx)$$

$$\therefore (b^2 - a^2) \int_0^P dx\, x J_n(ax) J_n(bx) = xW \Big|_0^P$$

$$= P\left[J_n(bP) J_n'(aP) - J_n(aP) J_n'(bP)\right]$$

Thus:

$$(b^2 - a^2) \int_0^P J_n(ax) J_n(bx)\, x\, dx = P\left[a J_n(bP) \frac{d}{d(ax)} J_n'(aP) - b J_n(aP) \frac{d}{d(bx)} J_n'(aP)\right]$$

To get identity ② Let $b = a + \epsilon$.

$$b^2 - a^2 = 2a\epsilon + \epsilon^2 \simeq 2a\epsilon \qquad \text{keeping first order in } \epsilon$$
$$J_n(bx) \simeq J_n(ax) + \epsilon x \, {}^{d}\!/_{d(ax)} J_n(ax)$$
$$J_n'(bx) \simeq J_n'(ax) + \epsilon x \, {}^{d}\!/_{d(ax)} J_n''(ax)$$

$$\therefore \text{LHS} = 2a\epsilon \int_0^P [J_n(ax)]^2 x\, dx \qquad \text{keeping first order in } \epsilon$$

Then

$$\text{RHS} = P\left[a\{J_n(aP) + \epsilon P J_n'(aP)\} J_n'(aP) - (a+\epsilon)\{J_n'(aP) + \epsilon P J_n''(aP)\} J_n(aP)\right]$$

$$= P\left[a\epsilon P [J_n'(aP)]^2 - a\epsilon P J_n(aP) J_n''(aP) - \epsilon J_n'(aP) J_n(aP)\right]$$

$$\equiv P^2 a\epsilon [J_n'(aP)]^2 - P\epsilon J_n(aP)\left(aP J_n''(aP) + J_n'(aP)\right)$$

$$= P^2 a\epsilon [J_n'(aP)]^2 + P\epsilon J_n(aP)\left(aP - \frac{n^2}{aP}\right) J_n(aP)$$

$$\therefore \quad \int_0^p \left(J_n(ax)\right)^2 x\,dx = \frac{p^2}{2}\left\{ \left[J_n'(ap)\right]^2 + \left(1 - \frac{n^2}{a^2p^2}\right)\left[J_n(ap)\right]^2 \right\}$$

Now $\quad J_n'(ap) = -J_{n+1}(ap) + \frac{n}{ap} J_n(ap)$

$\therefore$ Replace p by a and a by $\frac{\ell_{nm}}{a}$.

$$\int_0^a \left[J_n\left(\frac{\ell_{nm}}{a}x\right)\right]^2 x\,dx = \frac{a^2}{2}\left\{ \left[J_n'(\ell_{nm})\right]^2 + \left(1 - \frac{n^2}{\ell_{nm}^2}\right) J_n(\ell_{nm}) \right\}$$

since ℓ_{nm} is a zero of Bessel fn

$$= \frac{a^2}{2}\left[J_{n+1}(\ell_{n,m})\right]^2.$$