1. One Solution of
$$x^2y'' - 2y = 0, \qquad (x > 0)$$
on $0 < x < \infty$ is $y_1(x) = x^2$. Find all solutions of
$$x^2y'' - 2y = 2x - 1 \qquad (x > 0)\,.$$

Solution

Write in SL form
$$y'' - \frac{2}{x^2}y = 0.$$
Then $p(x) = 1$. Let $y_1 = x^2$. Thus
$$y_2(x) = y_1(x)\int \frac{dx}{p(x)\,[y_1(x)]^2} = -\frac{1}{3x}.$$
Now, Wronskian
$$W(y_1, y_2)(x) = y_1y_2' - y_2y_1' = 1.$$
Using method of variation of parameters, let $y_p = u_1y_1 + u_2y_2$. Let $R(x) = (2x-1)/x^2$. Then
$$u_1 = -\int \frac{y_2(x)R(x)dx}{W(x)} = -\frac{1}{6x^2}\,(4x-1)$$
And
$$u_2 = \int \frac{y_1(x)R(x)dx}{W(x)} = x^2 - x$$
Thus
$$y_p = \frac{1}{2}(1-2x)$$
Thus the general solution is
$$y_g = \frac{1}{2}(1-2x) + c_1x^2 + c_2\frac{1}{x}$$

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2. One solution of
$$x^2y'' - xy' + y = 0, \qquad (x > 0),$$
is $y_1(x) = x$. Find the solution ψ of
$$x^2y'' - xy' + y = x^2$$
satisfying $\psi(1) = 1$ and $\psi'(1) = 0$.

Solution

Put this in SL form by multiplying by $1/x^3$, then
$$\begin{aligned} \frac{1}{x}y'' - \frac{1}{x^2}y' + \frac{1}{x^3}y &= 0, \\ \frac{d}{dx}\left[\frac{y'}{x}\right] + \frac{1}{x^3}y &= 0. \end{aligned}$$
Now, $p(x) = 1/x$ and $R(x) = 1/x$. Given, $y_1 = x$, then
$$y_2(x) = y_1(x)\int \frac{dx}{p(x)\,[y_1(x)]^2} = x\ln x.$$

Wronskian $W(x) = x$. Writing $y_p = u_1 y_1 + u_2 y_2$, we get

$$\begin{aligned} u_1(x) &= -x\ln x + x \\ u_2(x) &= x \\ y_p(x) &= x^2 \end{aligned}$$

Thus general solution

$$y_g = c_1 x + c_2 x \ln x + x^2.$$

Applying BC, $y(1) = 1$ and $y'(1) = 1$, we get $c_1 = 0$ and $c_2 = -2$, that is

$$y(x) = x^2 - 2x\ln x.$$

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3. Show that a linear differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = p_3(x)\lambda y, \qquad (p_0(x) > 0),$$

can be transformed to SL differential equation by multiplying by

$$\frac{1}{p_0(x)} \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right].$$

Using this to transform Chebyshev II equation

$$\left(1 - x^2\right) y'' - 3xy' + n(n+2)y = 0, \qquad (x \in [-1, 1]),$$

to SL differential equation.

Solution

Let $P(x) = \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right]$. Then

$$P'(x) = \frac{p_1(x)}{p_0(x)} \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt\right]$$

Then, multiplying the differential equation by $P(x)/p_0(x)$, we get

$$P(x)y'' + P'(x)y' + \frac{p_2(x)P(x)}{p_0(x)} y = \frac{p_3(x)P(x)}{p_0(x)} y.$$

This is SL form.

Thus for Cheyshev II equation, $P(x) = (1 - x^2)^{3/2}$, that is multiply equation by $\sqrt{1 - x^2}$.

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4. Consider an eigenvalue problem

$$\begin{aligned} -\frac{d^2}{dx^2} y &= \lambda y, \qquad x \in [0, 1] \\ y(0) &= 0, \\ y'(1) &= y(1). \end{aligned}$$

   (a) Is this a Sturm-Liouville system? Is it regular, periodic or singular?
   (b) Find eigenvalues and eigenfunctions.
   (c) Verify that eigenfunctions constistute an orthogonal set. Normalize eigenfunctions suitably.

Solution

(a) Regular SL system.

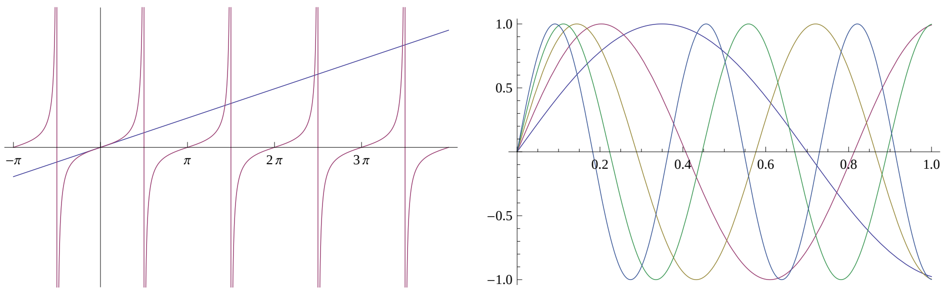
(b) General solution to the de is

$$y(x) = c_1 \sin\left(\sqrt{\lambda}x\right) + c_2 \cos\left(\sqrt{\lambda}x\right).$$

The BC $y(0) = 0$ implies that $c_2 = 0$ and BC $y(1) = y'(1)$ implies,

$$c_1 \sin\left(\sqrt{\lambda}\right) = c_1\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right).$$

Let t_n be nth solution of the equation $t = \tan t$. Then, the eigenvalues are $\lambda_n = t_n^2$ and eigenfunctions are $y_n = \sin(t_n x)$. The first few values of t_n are 4.49341, 7.72525, 10.9041, 14.0662, 17.2208 etc. The figure shows graphical way of obaining values of t_n. And it also shows the plots of first few eigenfunctions.



(a) (b)

Figure 1: Problem 4: Figure (a) shows the plots of t and $\tan t$. The intersections points are the solutions. Figure (b) shows the eigenfunctions.

(c) With bit of a trigonometry, it is easy to see that

$$\int_0^1 \sin(t_n x)\sin(t_m x)\,dx = 0$$

if $m \neq n$. And

$$\int_0^1 \sin^2(t_n x)\,dx = \frac{t_n^2}{2\left(1+t_n^2\right)}.$$

The eigenfunctions can be normalized by $\sqrt{2(1+t_n^2)}/t_n$.

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5. Consider an eigenvalue problem

$$\begin{aligned} -\frac{d^2}{dx^2}y &= \lambda y, \qquad x \in [0,b], \\ y(0) &= 0, \\ y'(0) &= y(b). \end{aligned}$$

(a) Is this a Sturm-Liouville system?

(b) Find all eigenvalues and eigenfunctions.

(c) Are eigenfunctions are orthogonal?

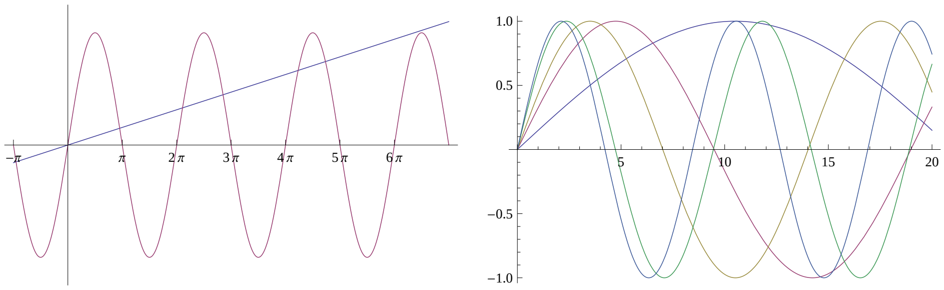
Solution

(a) Is not a SL system. The BC are mixed at two ends.

(b) Follow the same procedure as in previous problem, then we get eigenvalue and eigenfunctions as

$$\lambda_n = \frac{t_n^2}{b^2} \qquad y_n(x) = \sin\left(\frac{t_n x}{b}\right)$$

with t_n being nth positive root of $\sin t = t/b$. The figure below shows the solutions to this equation and the eigenfunctions for $b = 20$. There are only 5 eigenvalues.



(a) (b)

Figure 2: Figure (a) shows plots of t/b and $\sin t$. $b = 20$. Intersections are the solutions. Fig (b) Shows the actual eigenfunctions.

(c) Note:

$$\int_0^1 \sin(t_n x)\sin(t_m x)\,dx = \frac{t_m t_n}{t_m^2 - t_n^2}\left(\cos t_m - \cos t_n\right).$$

The left hand side is not necessarily zero. It is easy to see from the plots above.

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6. Show that the eigenvalues and eigenfunctions of the SL problem

$$\begin{aligned} \frac{d}{dx}\left[(1+x)^2\frac{dy}{dx}\right] + \lambda y &= 0, \qquad x \in [0,1], \\ y(0) = y(1) &= 0 \end{aligned}$$

are given by

$$\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 + \frac{1}{4}$$

and

$$y_n = \frac{1}{\sqrt{1+x}}\sin\left(n\pi\frac{\ln(1+x)}{\ln 2}\right).$$

Plot the first few functions and verify the theorem regarding the zeros of the eigenfunctions.

Solution

Verify eigenfunctions and eigenvalues by brute force. Here are the plots for the first few eigenfunctions.

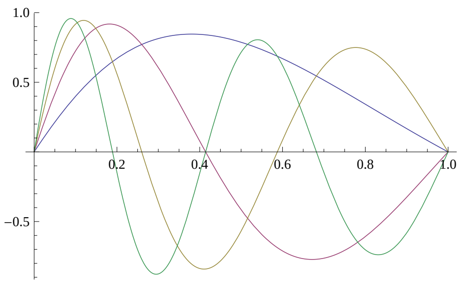

Figure 3: Problem 6
