1. Find two linearly independent power series solutions of the equation
$$y'' - xy' + y = 0.$$
For which values of x do the series converge?

2. Find a series solution for
$$(1+x^2)y'' + y = 0$$
about $x = 0$. [Is $P(x) = (1+x^2)^{-1}$ analytic everywhere?]

3. The equation
$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$
where α is a constant, is called the *Chebyshev equation*.
   (a) Compute two linearly independent series solutions for $|x| < 1$.
   (b) Show that for every non-negative integer $\alpha = n$ there is a polynomial solution of degree n. When appropriately normalized these are called *Chebyshev polynomials*.

4. The equation
$$y'' - 2xy' + 2\alpha y = 0$$
where α is a constant, is called *Hermite equation*.
   (a) Find two linearly independant series solutions for $-\infty < x < \infty$.
   (b) Are these solutions convergent for all x? Find the behaviour of these solutions for large x.
   (c) Show that there is a polynomial solution of degree n for every $\alpha = n$ non-negative integer.

5. Find the singular points of the following equations and determine those which are regular singular points:
   (a) $x^2y'' + (x+x^2)y' - y = 0$
   (b) $\left(1-x^2\right)y'' - 2xy' + l(l+1)y = 0$ (Legendre)
   (c) $x^2y'' - 5y' + 3x^2y = 0$
   (d) $\left(x^2+x-2\right)y'' + 3(x+2)y' + (x-1)y = 0$
   (e) $\left(1-x^2\right)y'' - xy' + n^2y = 0$ (Chebyshev)
   (f) $y'' - 2xy' + 2\alpha y = 0$ (Hermite)
   (g) $x^2y'' + xy' + (x^2-n^2)y = 0$ (Bessel)
   (h) $xy'' + (1-x)y' + ay = 0$ (Laguerre)

6. The equation
$$xy'' + (1-x)y' + ay = 0$$
where α is a constant, is called the *Laguerre equation*. Find two linearly independent series solutions about $x = 0$. Check convergence.

7. The interaction between two nucleons may be described by a mesonic potential
$$V(x) = \frac{Ae^{-ax}}{x}$$
where A and a are constants.Find the first few nonvanishing terms of the solution of 1D Schrodinger equation.

8. Find the series solutions of Legendre equation
$$\left(1-x^2\right)y'' - 2xy' + l(l+1)y = 0$$
about $x = 1$. Show that there are polynomial solutions for non-negative l.

Solutions

1. There are no singular points, thus let $y = \sum c_k x^k$. Substitute in the Differential equation

$$\sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} - x\sum_{k=0}^{\infty} kc_k x^{k-1} + \sum_{k=0}^{\infty} c_k x^k = 0$$

Adjusting the summation variable k in the first term to $k+2$, we get

$$\sum_{k=0}^{\infty} \left[(k+1)(k+2)c_{k+2} - (k-1)c_k\right] x^k = 0$$

Comparing the coefficients of x^k,

$$\left[(k+1)(k+2)c_{k+2} - (k-1)c_k\right] = 0$$

that is

$$c_{k+2} = \frac{(k-1)}{(k+1)(k+2)} c_k$$

Explicitly,

$$\begin{aligned} c_2 &= \frac{-1}{2\cdot 1}c_0 = -\frac{1}{2}c_0 \\ c_3 &= \frac{0}{3\cdot 2}c_1 = 0 \\ c_4 &= \frac{(-1)}{4\cdot 3}c_2 = \frac{(-1)(1)}{4!}c_0 = -\frac{1}{24}c_0 \\ c_6 &= \frac{3}{6\cdot 5}c_4 = \frac{(-1)\cdot 1\cdot 3}{6!}c_0 = -\frac{1}{240}c_0 \end{aligned}$$

Clearly, for odd $k \geq 3$, $c_k = 0$. For even k,

$$c_k = \frac{(-1)\cdot 1\cdots(k-3)}{k!}c_0$$

Thus

$$\begin{aligned} y &= c_0 + \sum_{k=2,4,\ldots} \frac{(-1)\cdot 1\cdots(k-3)}{k!}x^k + c_1 x \\ &= c_0\left[1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6\cdots\right] + c_1 x. \end{aligned}$$

The two solutions

$$\begin{aligned} y_1 &= \left[1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{240}x^6\cdots\right] \\ y_2 &= x \end{aligned}$$

are linearly independent and are convergent for all x. To see this,

$$\lim_{k\to\infty}\left|\frac{c_{k+2}x^2}{c_k}\right| = 0.$$

2. Substitute $y = \sum c_k x^k$.

$$\left(1+x^2\right)\sum_k k(k-1)c_k x^{k-2} + \sum_k c_k x^k = 0$$

Thus

$$\sum_k \left[(k+1)(k+2)\,c_{k+2} + (k(k-1)+1)\,c_k\right] x^k = 0$$

Thus, recurrence relation is

$$c_{k+2} = -\frac{(k(k-1)+1)}{(k+1)(k+2)} c_k.$$

Explicitly, for even k,

$$\begin{aligned} c_2 &= -\frac{1}{2\cdot 1} c_0 = -\frac{1}{2} c_0 \\ c_4 &= -\frac{3}{4\cdot 3} c_2 = +\frac{1\cdot 3}{4!} c_0 = \frac{1}{8} c_0 \\ c_6 &= -\frac{13}{6\cdot 5} c_4 = -\frac{1\cdot 3\cdot 13}{6!} c_0 = -\frac{13}{240} c_0 \end{aligned}$$

And for odd k,

$$\begin{aligned} c_3 &= -\frac{1}{3\cdot 2} c_1 = -\frac{1}{6} c_1 \\ c_5 &= -\frac{7}{5\cdot 4} c_3 = +\frac{1\cdot 7}{5!} c_1 \end{aligned}$$

Thus

$$y = c_0 \left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6 + \cdots\right] + c_1 \left[x - \frac{1}{6}x^3 + \frac{7}{120}x^5 + \cdots\right].$$

The two solutions are linearly independent and

$$\lim_{k\to\infty} \left|\frac{C_{k+2}x^2}{c_k}\right| = |x|^2$$

The series solutions converge only when $|x| < 1$. It is expected because $(1+x^2)^{-1}$ is analytic only for $|x| < 1$.

3. The Chebyshev equation is analytic at $x = 0$. (regular singular at ± 1). Thus substituting $y = \sum c_k x^k$ in the equation, we get recurrence relation

$$\begin{aligned} c_k &= \frac{\left((k-2)^2 - \alpha^2\right)}{k(k-1)} c_{k-2} \\ &= \frac{1}{k!}\left[\prod_{j=0,2}^{k-2} \left(j^2 - \alpha^2\right)\right] c_0 \qquad \text{even } k \\ &= \frac{1}{k!}\left[\prod_{j=1,3}^{k-2} \left(j^2 - \alpha^2\right)\right] c_0 \qquad \text{odd } k \end{aligned}$$

The two solutions are

$$\begin{aligned} y_1 &= 1 + \sum_{k=0,2,\ldots} \frac{1}{k!}\left[\prod_{j=0,2}^{k-2} \left(j^2 - \alpha^2\right)\right] x^k \\ y_2 &= x + \sum_{k=3,5,\ldots} \frac{1}{k!}\left[\prod_{j=1,3}^{k-2} \left(j^2 - \alpha^2\right)\right] x^k \end{aligned}$$

Clearly, series are convergent only for $|x| < 1$. And it is easy to see that if $\alpha = 2n$ some even integer then y_1 becomes a polynomial of degree $2n$ since $c_{2n+2} = 0$ and subsequently all further coefficients are zero. If $\alpha = 2n+1$, that is some odd integer, then y_2 would reduce to a polynomial.

4. The Hermite equation is analytic at all x. Again, substituting $y = \sum c_k x^k$, we get the recurrence relation

$$c_k = \frac{2\,(k-2-\alpha)}{k(k-1)} c_{k-2}.$$

For even $k = 2m$,

$$c_{2m} = \frac{2^m(2m-2-\alpha)\cdots(-\alpha)}{(2m)!} c_0$$

And for odd $k = 2m+1$

$$c_{2m+1} = \frac{2^m(2m-1-\alpha)\cdots(1-\alpha)}{(2m+1)!} c_1$$

Thus

$$y_1 = 1 - \alpha x + \frac{\alpha(\alpha-2)}{6} x^2 - \frac{\alpha(\alpha-2)(\alpha-4)}{90} x^6 + \cdots$$

and

$$y_2 = x - \frac{(\alpha-1)}{3} x^3 + \frac{(\alpha-1)(\alpha-3)}{30} x^5 + \cdots$$

Clearly,

$$\lim_{k\to\infty} \left| \frac{c_k x^2}{c_{k-2}} \right| = \lim_{k\to\infty} \left| \frac{2(k-2-\alpha)x^2}{k(k-1)} \right| = 0$$

Thus, both series converge for all x.

To obtain asymptotic behaviour, terms with large k matter, thus

$$c_{2m} = \frac{2}{2m} c_{2m-2} = \frac{1}{m!} c_0$$

Compare with

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \cdots + \frac{1}{(m)!}x^{2m} + \cdots$$

The series behaves like e^{x^2}.

It is easy to see that if $\alpha = n$ some integers either y_1 or y_2 becomes a polynomial.

5. skipped.

6. Given

$$xy'' + (1-x)y' + ay = 0$$

Note that $x = 0$ is a regular singular point. The indicial equation

$$r(r-1) + r + 0 = 0$$

has only one solution given by $r = 0$. The first solution is of the form $y = \sum_k c_k x^k$. Substitution gives us,

$$\begin{aligned} c_{k+1} &= \frac{(k-a)}{(k+1)^2} c_k \\ &= \frac{(k-a)\cdots(1-a)(-a)}{[(k+1)!]^2} c_0 \\ &= \frac{\Gamma(k+1-a)}{\Gamma(-a)\,[(k+1)!]^2} c_0 \end{aligned}$$

Thus

$$y_1 = \frac{c_0}{\Gamma(-a)} \left[1 - ax - \frac{a(1-a)}{4} x^2 - \cdots \right]$$

It is clear that the series converges for all x.
The second solution has the form given by

$$y = \sum_{k=0}^{\infty} d_k x^k + (\log x)\, y_1$$

with $d_0 = 0$. Substituting in the differential equation,

$$\begin{aligned} \sum_{k=0}^{\infty} \left[(k+1)^2 d_{k+1} + (a-k) d_k\right] x^k &= -2xy_1' \\ &= -2\sum k c_k x^k \end{aligned}$$

Thus,

$$\begin{aligned} d_1 &= 0 \\ 4d_2 &= -2c_1 \implies d_2 = \frac{a}{2} \\ 9d_3 + (a-2)d_2 &= -4c_2 \implies d_3 = -\frac{a(4-3a)}{18} \end{aligned}$$

Thus the second solution is

$$y_2 = \left[\frac{a}{2}x^2 - \frac{a(4-3a)}{18}x^3 + \cdots\right] + (\log x) y_1$$

7. Potential is given to be

$$V(x) = \frac{Ae^{-ax}}{x}$$

The Schrodinger equation is given by

$$\begin{aligned} -\frac{\hbar^2}{2m} y'' + (V(x) - E) y &= 0 \\ y'' + \left(\lambda^2 + \frac{B}{x} e^{-ax}\right) y &= 0 \qquad (1) \end{aligned}$$

where $\lambda^2 = 2mE/\hbar^2$ and $B = -2mA/\hbar^2$. Here, $x = 0$ is a regular singular point. The solution has a form

$$y = x^r \sum_{k=0}^{\infty} c_k x^k.$$

Substitute in equation 1:

$$\sum_{k=0}^{\infty} (r+k)(r+k-1) c_k x^{r+k-2} + \sum_{k=0}^{\infty} \left[\lambda^2 + B \sum_{i=0}^{\infty} \frac{x^{i-1}}{i!}\right] c_k x^{r+k} = 0$$

Consider the double sum, put $i + k = m$ and replace k sum by m sum

$$\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{c_k}{i!} x^{r+k+i-1} = \sum_{m=0}^{\infty} \left(\sum_{i=0}^{m} \frac{c_{m-i}}{i!}\right) x^{m+r-1}$$

And the for the first sum put $k - 2 = m$, thenx

$$r(r-1)c_0 x^{r-2} + ((r+1)rc_1 + Bc_0)\, x^{r-1} \sum_{m=1}^{\infty} \left[(r+m+1)(r+m)c_{m+1} + \lambda^2 c_{m-1} + B\left(\sum_{i=0}^{m} \frac{c_{m-i}}{i!}\right)\right] x^{m+r-1} =$$

Indicial equation is

$$r(r-1) = 0.$$

putting $r = 1$, we get the first solution

$$\begin{aligned} 2c_1 + Bc_0 &= 0 \implies c_1 = -\frac{B}{2}c_0 \\ 3 \cdot 2 \cdot c_2 + \lambda^2 c_0 + B\left(c_0 + c_1\right) &= 0 \implies c_0 = -\frac{1}{6}\left(\lambda^2 + B\left(1 - \frac{B}{2}\right)\right)c_0 \end{aligned}$$

Thus the solution is

$$y = c_0 x\left[1 - \frac{B}{2}x - \frac{1}{6}\left(\lambda^2 + B\left(1 - \frac{B}{2}\right)\right)x^2 + \cdots\right]$$

8. Legendre Equation

$$\left(1-x\right)\left(1+x\right)y'' - 2xy' + l(l+1)y = 0$$

Make a substitution $z = 1 - x$. Then $dy/dx = -dy/dz$. Legendre equation becomes

$$z\left(2-z\right)y'' + 2\left(1-z\right)y' + l\left(l+1\right)y = 0$$

where the derivatives are wrt z. Clearly $z = 0$ and 2 are regular singular points. Using Frobenius method with power series about $z = 0$. Let

$$y = z^r \sum c_k z^k.$$

Indicial equation is

$$r(r-1) + r = 0 \implies r = 0.$$

The regular solution is given by $y = \sum c_k z^k$. Substituting in the de, we get

$$\begin{aligned} c_1 &= -\frac{l(l+1)}{2}c_0 \\ c_{k+1} &= \frac{k(k+1) - l(l+1)}{2(k+1)^2}c_k \qquad k > 0. \end{aligned}$$

Thus for $l = 0$, $c_k = 0$ for all $k > 0$

$$y = c_0$$

For $l = 1$, $c_1 = -c_0$ and $c_k = 0$ for all $k > 1$

$$y = c_0(1-z) = c_0\left[1 - (1-x)\right] = c_0 x$$

For $l = 2$, $c_1 = -3c_0$, $c_2 = 3c_0/2$, so

$$y = c_0\left(1 - 3z + \frac{3}{2}z^2\right)$$

Finally, for $l = 3$, we get

$$y = c_0\left(1 - 6z + \frac{15}{2}z^2 - \frac{5}{3}z^3\right)$$