1. Solve:

   (a) $e^{x^2+y}\,dx + \frac{y}{2x}\,dy = 0$
   (b) $xy^3\frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$
   (c) $x^2\,(y+1)\,dx + y^2(x-1)\,dy = 0$

2. Solve:

   (a) $[\cos x \tan y + \cos(x+y)]\,dx + [\sin x \sec^2 y + \cos(x+y)]\,dy = 0$
   (b) $(3x^2+4y)\,dx + (4x - y + 1)\,dy = 0$
   (c) $\frac{dy}{dx} = \frac{y+1}{(y+2)e^y - x}$

3. Solve:

   (a) $(x^2+y^2+2x)\,dx + 2y\,dy = 0$
   (b) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

4. Solve:

   (a) $y' + \frac{y}{x} = 4\,(1+x^2)$
   (b) $y' + x\sin 2y = x^3\cos^2 y$
   (c) $(x+2y^3)\,y' = y$
   (d) $\left(xy - \frac{dy}{dx}\right)e^{x^2} = y^3$

5. Find general solutions for each of the following equations.

   (a) $y'' + y' - 6y = 0$
   (b) $y'' - 4y' + 4y = 0$
   (c) $y'' - 4y = 0$
   (d) $y'' - 5y' = 0$

6. Use the method of undetermined coefficients to solve each of the following:

   (a) $y'' + y = x^2 + 2x$
   (b) $y'' + 10y' + 25y = 20e^{-5x}$
   (c) $y'' + y = x\sin x$

7. Use the method of variation of parameters to find solutions for each of the following:

   (a) $y'' + 4y = \tan 2x$
   (b) $y'' + 4y' + 5y = e^{-2x}\sec x$
   (c) $y'' + y = \frac{1}{1+\sin x}$

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Q1. Variable Separation:

(a) $$e^{x^2+y}\,dx + \frac{y}{2x}\,dy = 0$$

$$\Rightarrow\ 2x e^{x^2} dx + y e^{-y} dy = 0$$

$$\Rightarrow\ \int 2x e^{x^2} dx + \int y e^{-y} dy = c$$ c: some constant.

$$\Rightarrow\ e^{x^2} + (-1)\cdot(y+1)e^{-y} = c.$$

$$\Rightarrow\ (y+1)e^{-y} = e^{x^2} - c.$$

(b) $$x y^3 \frac{dy}{dx} = 1 - x^2 + y^2 - x^2y^2$$

$$= (1-x^2)(1+y^2)$$

$$\Rightarrow\ \frac{y^3 dy}{1+y^2} = \frac{(1-x^2)\,dx}{x}$$

$$\Rightarrow\ \frac{y^2}{2} - \frac{1}{2}\ln(1+y^2) = \ln x - \frac{x^2}{2} + c.$$

(c) $$x^2(y+1)\,dx + y^2(x-1)\,dy = 0$$

$$\Rightarrow\ \frac{x^2\,dx}{(x-1)} + \frac{y^2\,dy}{(1+y)} = 0$$

$$\Rightarrow\ x + \frac{x^2}{2} + \ln(x-1) - y + \frac{y^2}{2} + \ln(1+y) = c$$

Q2. Exact Differentials.

(a) $$\left(\cos x \tan y + \cos(x+y)\right)dx + \left(\sin x \sec^2 y + \cos(x+y)\right)dy = 0$$

Easy to check that this is an exact DE.

$M = \cos x \tan y + \cos(x+y)$ $\quad \frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x+y)$

$N = \sin x \cdot \sec^2 y + \cos(x+y)$ $\quad \frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x+y)$

Soln. is

$$\int\left(\cos x \tan y + \cos(x+y)\right)dx = C$$

$$\Rightarrow\ \sin x \tan y + \sin(x+y) = c.$$

Q2.

(b) $\underbrace{(3x^2+4y)}_{M}\,dx + \underbrace{(4x-y+1)}_{N}\,dy = 0$

Then $\dfrac{\partial M}{\partial y} = 4$ and $\dfrac{\partial N}{\partial x} = 4 \Rightarrow$ DE is exact.

Sol^n is.

$$\int (3x^2+4y)\,dx + \int (-y+1)\,dy = c.$$

$$\Rightarrow x^3 + 4xy - \frac{y^2}{2} + y = c.$$

(c) $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)e^y - x}$

$\Rightarrow (y+1)\,dx + (x - (y+2)e^y)\,dy = 0$

$\Rightarrow$ Exact DE

Sol^n is

$$x(y+1) - (y+1)e^y = c.$$

$$(y+1)(x-e^y) = c.$$

Q3. Using Integrating factors.

(a) $\underbrace{(x^2+y^2+2x)}_{M}\,dx + \underbrace{2y}_{N}\,dy = 0.$

$\Rightarrow \dfrac{\partial M}{\partial y} = 2y$ and $\dfrac{\partial N}{\partial x} = 0$ $\quad \dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = 1 = f(x)$

$\Rightarrow IF = e^x. = e^{\int f(x)dx}$

Thus: $e^x(x^2+y^2+2x) + 2e^x y\,dy = 0$ is exact.

$\Rightarrow$ Sol^n is

$$\int e^x(x^2+y^2+2x)\,dx = C$$

$$\Rightarrow e^x(x^2+y^2) = c.$$

Q3.

(b) $$\underbrace{\left(3x^2y^4+2xy\right)}_{M} dx + \underbrace{\left(2x^3y^3 - x^2\right)}_{N} dy = 0.$$

Check: $\left(M_x + N_y = 5x^3y^4\right)$ No!

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{(3x^2y^4+2xy)}\left[6x^2y^3 - 2x - 12x^2y^3 - 2x\right]$$

$$= -\frac{2}{y}$$

$$\Rightarrow IF = \exp\left[\int\left(-\frac{2}{y}\right)dy\right] = +\frac{1}{y^2}.$$

<u>That is</u>

$$\frac{1}{y^2}\left(3x^2y^4+2xy\right)dx + \frac{1}{y^2}\left(2x^3y^3 - x^2\right)dy = 0$$

is exact.

Sol$^{\underline{n}}$ is

$$\int \frac{1}{y^2}\left(3x^2y^4+2xy\right)dx + 0 = c.$$

$$\Rightarrow \frac{1}{y^2}\left(x^3y^4 + x^2y\right) = c.$$

Q4. Linear Equations.

(a) $$y' + \frac{1}{x}y = 4(1+x^2)$$

Let $I(x) = \int \frac{1}{x}dx = \ln x \quad \Rightarrow e^{I(x)} = \frac{1}{x}$

Sol$^{\underline{n}}$ is

$$y = x\int \frac{1}{x}\cdot 4(1+x^2)\,dx + cx.$$

$$= 4x\left(\ln x + \frac{x^2}{2}\right) + cx$$

Q4(b) $y' + x \sin 2y = x^3 \cos^2 y$

Now put $z = \tan y \Rightarrow \dfrac{dz}{dx} = \sec^2 y \dfrac{dy}{dx}$.

$$\Rightarrow \frac{1}{\sec^2 y}\frac{dz}{dx} + x\, 2\sin y \cos y = x^3 \cos^2 y$$

$$\Rightarrow \frac{dz}{dx} + 2x\, z = x^3$$

$\Rightarrow$ Let $I(x) = \int 2x\,dx = x^2 \Rightarrow e^{I(x)} = e^{x^2}$

Sol^n

$$\Rightarrow \quad z = e^{-x^2}\int e^{x^2}\cdot x^3 dx + e^{-x^2}\cdot c$$

$$\Rightarrow \tan y = \frac{1}{2}(x^2-1) + c\, e^{-x^2}.$$

(c) $(x + 2y^3)\dfrac{dy}{dx} = y$

$$\Rightarrow y\frac{dx}{dy} = x + 2y^3$$

$$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

$\Rightarrow$ Let $I(x) = -\int \frac{1}{y}dy = -\ln y$, $e^{I(x)} = e^{-\ln y} = \frac{1}{y}$.

The Sol^n is

$$x = y\int \frac{1}{y}\cdot(2y^2)\,dy + cy$$

$$= y^3 + cy.$$

(d) $\left(xy - \dfrac{dy}{dx}\right)e^{+x^2} = y^3$

$$\Rightarrow \frac{dy}{dx} - xy = -e^{-x^2}y^3.$$ Bernoullies Eq.

put: $z = \dfrac{1}{y^2} \Rightarrow \dfrac{dz}{dx} = -\dfrac{2}{y^3}\dfrac{dy}{dx}$

$$\Rightarrow -\frac{y^3}{2}\frac{dz}{dx} - xy = -e^{-x^2}y^3$$

$$\Rightarrow \frac{dz}{dx} + \frac{2x}{y^2} = e^{-x^2}$$

$$\Rightarrow \frac{dz}{dx} + 2x\,z = e^{-x^2}. \Rightarrow z = x e^{-x^2} + c e^{-x^2}$$

$$\Rightarrow \frac{1}{y^2} = x e^{-x^2} + c e^{-x^2}.$$

Q5 (a) $y'' + y' - 6y = 0$

Substituting $y = e^{px}$ we get

$$p^2 + p - 6 = 0 \Rightarrow p = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+24} = -\frac{1}{2} \pm \frac{5}{2}$$

$$= 2 \text{ or } -3.$$

$$\therefore y_g = c_1 e^{2x} + c_2 e^{-3x}.$$

(b) $y'' - 4y' + 4y = 0$

$$y_g = (c_1 + c_2 x) e^{2x}$$

(c) $y'' - 4y = 0$

$$y_g = c_1 e^{2x} + c_2 e^{-2x}$$

(d) $y'' - 5y' = 0$

$$y_g = c_1 + c_2 e^{5x}$$

Q6. (a) Given $y'' + y = x^2 + 2x$ — (1)

Now general sol^n to $y'' + y = 0$ is $y = c_1 \sin x + c_2 \cos x$.

To find perticular sol^n: The UDC set $= \{x^2, x, 1\}$

Let
$$\left.\begin{aligned} y_p &= a + bx + cx^2 \\ y_p' &= b + 2cx \\ y_p'' &= 2c \end{aligned}\right\} \text{Substitute in (1)}$$

$$2c + a + bx + cx^2 = x^2 + 2x$$

$$\Rightarrow b = 2, \quad c = 1, \quad a = -2$$

$$\Rightarrow y = c_1 \sin x + c_2 \cos x + (x^2 + 2x - 2)$$

Q6 (b) $\quad y'' + 10y' + 25y = 20e^{-5x}$

General soln to $y'' + 10y' + 25y = 0$ is

$$y_g = c_1 e^{-5x} + c_2 x e^{-5x}$$

UDC set $= \{e^{-5x}\}$, since e^{-5x} is also soln of homogeneous DE, UDC set $= \{x^2 e^{-5x}\}$

Let $y_p = A x^2 e^{-5x}$

$$y_p' = A\left[2x - 5x^2\right]e^{-5x}$$

$$y_p'' = A\left[2 - 10x - 10x + 25x^2\right]e^{-5x}$$

$$\Rightarrow \quad A\left[2 - \cancel{20x} + \cancel{25x^2} + \cancel{20x} - \cancel{50x^2} + \cancel{25x^2}\right] = 20$$

$$\Rightarrow \quad A = 10.$$

Thus $y_g = 10x^2 e^{-5x} + (c_1 + c_2 x)e^{-5x}$.

Q 6 (c) $y'' + y = x \sin x$

Homogeneous sol^n $\quad y_g = C_1 \sin x + C_2 \cos x$.

UDC Set $= \{ x \sin x,\ x \cos x,\ \sin x,\ \cos x \}$

Multiply By x since $\sin x$ and $\cos x$ repeat in Homogenous Sol^n.

Thus guess for y_p

$$y_p = (Ax^2 + Bx) \sin x + (Cx^2 + Dx) \cos x$$

$$y_p' = (Ax^2 + Bx) \cos x + (2Ax + B) \sin x$$
$$- (Cx^2 + Dx) \sin x + (2Cx + D) \cos x$$

$$y_p'' = (Ax^2 + (B + 2C)x + D)\ \cancel{\cos x}(-\sin x)$$
$$+ (2Ax + (B + 2C)) \cos x$$
$$+ (2Ax \pm Dx - Cx^2 + B)\ \cancel{\sin}\ \cos x$$
$$+ (-2Cx + (2A - D)) \sin x.$$

$$\therefore y_p'' + y_p = \cos x \left[\cancel{Ax^2} - Cx^2 + (4A - D)x + 2B + 2C \right]$$
$$+ \sin x \left[-Ax^2 + (-B - 4C)x + (2A - 2D) \right]$$
$$+ (Ax^2 + Bx) \sin x + (Cx^2 + Dx) \cos x$$
$$= \cancel{x \sin x}.$$
$$= x \sin x.$$

$\Rightarrow$

$\Rightarrow$ $-4C = 1$ $\quad 2A - 2D = 0$

$4A = 0$ $\quad 2B + 2C = 0$

$A = 0$, $C = -\frac{1}{4}$, $B = \frac{1}{4}$, $D = 0$.

$$y_p = -\frac{1}{4} x^2 \cos x + \frac{1}{4} x \sin x.$$

Q7 (a) $y'' + 4y = \tan 2x.$

- Homogeneous sol^{n}: $y_1 = \sin(2x)$ and $y_2 = \cos(2x)$

- Let $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$.

then $u_1(x) = -\int \frac{\cos(2x) \cdot \tan(2x)}{(-2)}\, dx$, since $W(y_1, y_2) = -2$.

$$= +\frac{1}{2}\int \sin(2x)\, dx = \frac{-\cos(2x)}{4}$$

and $u_2(x) = +\int \frac{\sin(2x)\tan(2x)}{(-2)}\, dx$

$$= +\frac{1}{2}\int \left[\cos(2x) - \sec(2x)\right] dx$$

$$= +\frac{1}{2}\left[\frac{\sin(2x)}{2} - \frac{1}{2}\ln\left[\sec(2x) + \tan(2x)\right]\right]$$

Thus $y_p = -\frac{1}{4}\sin(2x)\cancel{\cos(2x)} + \frac{1}{4}\cancel{\sin(2x)}\cos(2x)$
$- \frac{1}{4}\ln\left[\sec(2x) + \tan(2x)\right] \cdot \cos(2x)$

Thus GS to DE

$$y = C_1 \sin(2x) + C_2\cos(2x) - \frac{\cos(2x)}{4}\ln\left[\sec(2x) + \tan(2x)\right].$$

(b) $y'' + 4y' + 5y = e^{-2x}\sec x.$

Homogenous sol^{n}: $y_1 = e^{-2x}\sin x$ and $y_2 = e^{-2x}\cos(2x)$

[Note: $p^2 + 4p + 5 = 0 \Rightarrow p_1 = -2 \pm i$

So two sol^{n} are $e^{(-2+i)x}$ and $e^{(-2-i)x}$

Now y_1 and y_2 given above are LC of these]

$W(y_1, y_2) = -e^{-4x}.$

Then $$u_1(x) = -\int \frac{e^{-2x}\cos(x) \cdot e^{-2x}\sec(x)}{-e^{-4x}}\,dx$$

$$= x$$

and $$u_2(x) = \int \frac{e^{-2x}\sin(x) \cdot e^{-2x}\sec(x)}{-e^{-4x}}\,dx$$

$$= -\int \tan(x)\,dx = -\ln(\sec(x)) = \ln(\cos(x))$$

$$\therefore\ y = e^{-2x}\left(c_1 \sin x + c_2 \cos x\right)$$

$$+ x\sin(x) + \cos(x)\ln\cos(x)$$

(c) $$y'' + y = \frac{1}{1+\sin x}$$

Homogeneous solⁿ ~~$y_1 = e^{-ix}$ $y_2 = e^{+ix}$~~

$$y_1 = \sin x \qquad y_2 = \cos x \qquad W(y_1, y_2) = 1$$

Then $$u_1 = -\int \frac{\cos x}{(1+\sin x)}\,dx$$

$$= -\ln(1+\sin x)$$

and $$u_2 = \int \frac{\sin x}{1+\sin x}\,dx$$

$$= \int \left[1 - \frac{1}{1+\sin x}\right] dx$$

$$= x - \int \frac{\sec^2(\frac{x}{2})}{[1+\tan(\frac{x}{2})]^2}\,dx$$

$$= x - \frac{2\cos(\frac{x}{2})}{\cos(\frac{x}{2}) + \sin(\frac{x}{2})}.$$

$\therefore$ Solⁿ.

$$y = c_1 \sin x + c_2 \cos x - \sin(x)\ln(1+\sin x)$$

$$+ \cos x\left(x - \frac{2\cos(x/2)}{\cos(x/2) + \sin(x/2)}\right).$$