1 Sturm-Liouville System

1.1 Sturm-Liouville Differential Equation

Definition 1. A second ordered differential equation of the form

$$-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y + q(x)y = \lambda\omega(x)y, \qquad x \in [a,b] \tag{1}$$

with p, q and ω specified such that $p(x) > 0$ and $\omega(x) > 0$ for $x \in (a,b)$, is called a *Sturm-Liouville(SL) differential equation.*

Note that SL differential equation is essentially an eigenvalue problem since λ is not specified. While solving SL equation, both λ and y must be determined.

Example 2. The Schrodinger equation

$$-\frac{\hbar^2}{2m}\psi + V(x)\psi = E\psi$$

on an interval $[a,b]$ is a SL differential equation.

1.2 Boundary Conditions

Boundary conditions for a solution y of a differential equation on interval $[a,b]$ are classified as follows:

Mixed Boundary Conditions Boundary conditions of the form

$$\begin{aligned} c_a y(a) + d_a y'(a) &= \alpha \\ c_b y(b) + d_b y'(b) &= \beta \end{aligned} \tag{2}$$

where, $c_a, d_a, c_b, d_b, \alpha$ and β are constants, are called *mixed Dirichlet-Neumann* boundary conditions. When both $\alpha = \beta = 0$ the boundary conditions are said to be homogeneous. Special cases are *Dirichlet* BC ($d_a = d_b = 0$) and *Neumann* BC ($c_a = c_b = 0$)

Periodic Boundary Conditions Boundary conditions of the form

$$\begin{aligned} y(a) &= y(b) \\ y'(a) &= y'(b) \end{aligned} \tag{3}$$

are called *periodic* boundary conditions.

Example 3. Some examples are

1. Stretched vibrating string clamped at two ends: Dirichlet BC
2. Electrostatic potential on the surface of a volume: Dirichlet BC
3. Electrostatic field on the surface of a volume: Neumann BC
4. A heat-conducting rod with two ends in heat baths: Dirichlet BC

1.3 Sturm-Liouville System (Problem)

Definition 4. The SL differential equation on a finite interval $[a, b]$ with homogeneous mixed boundary conditions, that is,

$$\begin{aligned}
-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y + q(x)y &= \lambda\omega(x)y, \qquad x \in [a, b] \\
c_a y(a) + d_a y'(a) &= 0 \\
c_b y(b) + d_b y'(b) &= 0
\end{aligned}$$

with $p(x) > 0$ and $\omega(x) > 0$ for $x \in [a, b]$ is called as *regular Sturm-Liouville system* (or problem).

Aim is to find all values λ for which a nontrivial solution y_λ exists. It is implicitly assumed that y_λ and its derivative are continuous on $[a, b]$, which also means these are bounded.

Example 5. Quantum particle in a 1D box: The Schrodinger equation and boundary conditions are given by

$$\begin{aligned}
-\frac{\hbar^2}{2m}\psi(x) &= E\psi(x) \qquad x \in [0, L] \\
\psi(0) &= 0 \\
\psi(L) &= 0.
\end{aligned}$$

This is a regular Sturm-Liouville system. The eigenvalues and eigenfunctions are

$$\begin{aligned}
E_n &= \frac{\hbar^2\pi^2 n^2}{2mL^2} \\
\psi_n(x) &= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

with $n = 1, 2, \ldots$.

Definition 6. A SL differential equation on an interval $[a, b]$ with periodic boundary conditions and $p(a) = p(b)$ is called as *periodic Sturm-Liouville system.*

Example 7. Quantum particle freely moving on a circle. The Schrodinger equation is

$$\begin{aligned}
-\frac{\hbar^2}{2m}\psi(x) &= E\psi(x) \qquad x \in [0, L] \\
\psi(0) &= \psi(L) \\
\psi'(0) &= \psi'(L)
\end{aligned}$$

This is an example of a periodic SL system. The eigenvalues and eigenfunctions are

$$E_n = \frac{4\pi^2\hbar^2 n^2}{2mL^2} \qquad n = 0, 1, \ldots$$

and for each E_n (except $n = 0$) there are two linearly independant eigenfunctions

$$\cos\left(\frac{2\pi n x}{L}\right), \qquad \sin\left(\frac{2\pi n x}{L}\right).$$

Definition 8. A SL differential equation on an interval $[a, b]$ with any of the following conditions will be called a *singular Sturm-Liouville system.*

1. $p(a) = 0$, BC at a is dropped, BC at b is homogenous mixed.
2. $p(b) = 0$, BC at b is dropped, BC at a is homogenous mixed.

3. $p(b) = p(a) = 0$ and no BC.

4. Interval $[a, b]$ is infinite.

Note:

1. If $p(a) = 0$ and there is no BC at a, the y is considered a solution if $y(a) < \infty$. Simillarly for the other cases.

2. If the interval is infinite, then y must be square integrable to be considered as a solution.

Example 9. Here are few examples.

1. Legendre Equation is given by

$$\left(1-x^2\right)y'' - 2xy' + l(l+1)y = 0 \quad x \in [-1, 1]$$

This can be cast in the form

$$-\frac{d}{dx}\left[\left(1-x^2\right)y'\right] = l(l+1)y.$$

Here $p(x) = 1 - x^2$, $q(x) = 0$, $\omega(x) = 0$ and $\lambda = l(l+1)$. However since $p(-1) = p(1) = 0$, this is a singular SL problem.

2. Chebyshev equation is given by

$$\left(1-x^2\right)y'' - xy' + n^2 y = 0 \quad x \in [-1, 1]$$

This can be converted to SL form by dividing by $\sqrt{1-x^2}$:

$$\begin{aligned} \sqrt{1-x^2}y'' - \frac{x}{\sqrt{1-x^2}}y' + \frac{n^2}{\sqrt{1-x^2}}y &= 0 \\ -\frac{d}{dx}\left[\frac{1}{\sqrt{1-x^2}}y'\right] &= \frac{n^2}{\sqrt{1-x^2}}y \end{aligned}$$

Here $p(x) = \left(1-x^2\right)^{-1/2}$, $q(x) = 0$, $\omega(x) = \left(1-x^2\right)^{-1/2}$ and $\lambda = n^2$. This is a singular SL System.

3. Hermite equation is given by

$$y'' - 2xy' + 2\alpha y = 0 \qquad x \in (-\infty, \infty)$$

This equation can be cast in the SL form by multiplying it by e^{-x^2}. Then,

$$-\frac{d}{dx}\left[e^{-x^2}y'\right] = 2\alpha e^{-x^2}y.$$

Here $p(x) = e^{-x^2}$, $q(x) = 0$, $\omega(x) = e^{-x^2}$ and $\lambda = 2\alpha$. This is a singular SL System because interval is infinite.

4. Finally, Laguerre equation

$$xy'' + (1-x)y' + ay = 0 \qquad x \in [0, \infty)$$

can be converted to SL form by multiplying e^{-x}. Then,

$$-\frac{d}{dx}\left[xe^{-x}y'\right] = ae^{-x}y.$$

Here $p(x) = xe^{-x}$, $q(x) = 0$, $\omega(x) = e^{-x}$ and $\lambda = a$. This is a singular SL System because interval is infinite and also $p(0) = 0$.

2 Properties of Sturm-Liouville System

It is interesting to note that a lot of information about the eigenvalues and eigenfunctions can be obtained without actually solving the SL problem. Some properties are that the eigenvalues are always real and bounded below but not above. If the interval $[a, b]$ is finite, then eigenvalues are discrete. Eigenfunctions are oscillatory in nature, and so on. We will set to prove some of these properties in the following sections.

2.1 Sturm-Liouville Operator

Consider a regular SL problem

$$\begin{aligned} -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]y + q(x)y &= \lambda\omega(x)y, \qquad x \in [a, b] \\ c_a y(a) + d_a y'(a) &= 0; \\ c_b y(b) + d_b y'(b) &= 0. \end{aligned}$$

Let $\mathcal{L}^2([a, b], \omega(x), dx)$ be the Hilbert space of square integrable functions on $[a, b]$ with inner product

$$\langle f, g\rangle = \int_a^b \overline{f(x)}g(x)\,\omega(x)dx$$

with $\omega(x)$ called *weight function*. Let $\mathcal{H}$ be the subspace of functions that satisfy the boundary conditions of SL problem. Now, the differential operator of the form

$$L = \frac{1}{\omega(x)}\left[-\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right]$$

on some domain in $\mathcal{H}$, is called a *Sturm-Liouville operator*. Then the SL differential equation becomes an eigenvalue equation in the space H

$$Ly = \lambda y.$$

Theorem 10. *Sturm-Liouville operator is self-adjoint operator on $\mathcal{H}$.*

Proof. Note

$$\begin{aligned} \langle f, Lg\rangle &= \int_a^b \overline{f(x)}\,(Lg)\,(x)\,\omega(x)dx \\ &= \int_a^b f(x)\left[-\frac{d}{dx}\left[p(x)g'(x)\right] + q(x)g(x)\right]dx \\ &= \end{aligned}$$

Integrating the first term by parts

$$\langle f, Lg\rangle = -\left[p(x)\overline{f(x)}g'(x)\right]_a^b + \int_a^b \left[\overline{f'(x)}p(x)g'(x) + \overline{f(x)}q(x)g(x)\right]dx$$

Simillary,

$$\langle f, Lg\rangle = -\left[p(x)\overline{f'(x)}g(x)\right]_a^b + \int_a^b \left[\overline{f'(x)}p(x)g'(x) + \overline{f(x)}q(x)g(x)\right]dx$$

Thus

$$\begin{aligned} \langle f, Lg\rangle - \langle Lf, g\rangle &= -\left[p(x)\overline{f(x)}g'(x)\right]_a^b + \left[p(x)\overline{f'(x)}g(x)\right]_a^b \\ &= p(b)\left(\overline{f'(b)}g(b) - \overline{f(b)}g'(b)\right) - p(a)\left(\overline{f'(a)}g(a) - \overline{f(a)}g'(a)\right) \qquad (4) \end{aligned}$$

Now, since both f, and g obey same boundary conditions,

$$\begin{aligned} c_a f(a) + d_a f'(a) &= 0 \\ \implies c_a \overline{f(a)} + d_a \overline{f'(a)} &= 0 \\ \text{and} \quad c_a g(a) + d_a g'(a) &= 0 \end{aligned}$$

it is easy to see that

$$\left(\overline{f'(a)}g(a) - \overline{f(a)}g'(a)\right) = 0$$

if $c_a \neq 0$ or $d_a \neq 0$. Simillary,

$$\left(\overline{f'(b)}g(b) - \overline{f(b)}g'(b)\right) = 0$$

Hence

$$\langle f, Lg \rangle = \langle Lf, g \rangle$$

□

Now, this result can easily be extended to periodic and singular SL systems. The two changes in the proof will be as follows:

- ▷ The subspace $\mathcal{H}$ will be appropriately defined by the BC.
- ▷ Note that rhs of the equation 4 will still be zero, if
  - $p(a) = p(b)$ and boundary conditions are periodic (equation 3) (That is if SL system is periodic.);
  - $p(a) = 0$ and boundary condition at b is homogeneous (Singular SL system);
  - $p(b) = 0$ and boundary condition at a is homogeneous (Singular SL system);
  - interval $[a, b]$ is infinite (since at infinity functions will be vanishing) (Singular SL systems).

2.2 Properties of SL systems

Theorem 11. *Eigenvalues of Sturm-Liouville problem are real.*

Proof. Let y_λ be an eigenfunction corresponding to eigenvalue λ. Then

$$Ly_\lambda(x) = \lambda y_\lambda(x).$$

Now,

$$\begin{aligned} \langle Ly_\lambda, y_\lambda \rangle &= \langle y_\lambda, Ly_\lambda \rangle \\ \therefore \overline{\lambda} \langle y_\lambda, y_\lambda \rangle &= \lambda \langle y_\lambda, y_\lambda \rangle \end{aligned}$$

Since $\langle y_\lambda, y_\lambda \rangle \neq 0$, $\overline{\lambda} = \lambda$. □

Theorem 12. *If λ_m and λ_n are two distinct eigenvalues of a SL system, with corresponding eigenfunctions y_m and y_n, then y_m and y_n are orthogonal.*

Proof. Note

$$\begin{aligned} \langle Ly_m, y_n \rangle &= \langle y_m, Ly_n \rangle \\ \therefore \lambda_m \langle y_m, y_n \rangle &= \lambda_n \langle y_m, y_n \rangle \\ \therefore (\lambda_m - \lambda_n) \langle y_m, y_n \rangle &= 0. \end{aligned}$$

Since $(\lambda_m - \lambda_n) \neq 0$, $\langle y_m, y_n \rangle = 0$ □

Theorem 13. *Eigenvalues of a regular SL system are non-degenerate (that is, there is a unique eigenfunction upto a constant).*

Proof. Let y_1 and y_2 are eigenfunctions corresponding to the given eigenvalue λ. Then,

$$\begin{aligned} Ly_1 &= \lambda y_1 \\ Ly_2 &= \lambda y_2. \end{aligned}$$

Now,

$$\begin{aligned} y_2(x)Ly_1(x) - y_1(x)Ly_2(x) &= 0 \\ -y_2(x)\frac{d}{dx}\left[p(x)y_1'\right] + y_1(x)\frac{d}{dx}\left[p(x)y_2'\right] &= 0 \\ \frac{d}{dx}\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right] &= 0 \end{aligned}$$

That is

$$p(x)W\left(y_1, y_2\right)(x) = \text{constant} = c$$

for all $x \in [a, b]$. However, the Wronskian of these functions

$$W(y_1, y_2)(a) = y_1(a)y_2'(a) - y_1'(a)y_2(a) = 0$$

because y_1 and y_2 satisfy the same boundary conditions at a. Thus,

$$W(y_1, y_2)(x) = 0$$

for all x. Then the two functions must be linearly dependent. That is $y_1 \propto y_2$. □

There is an additional benefit in the proof. This gives us a way of calculating the second LI solution if we know one solution. Let y_1 be one of the known solution of the differential equation $Ly = \lambda y$. Then, for another solution y_2, which is linearly independent of y_1,

$$p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right) = c$$

holds true. Then

$$p(x)\left(y_1(x)\right)^2 \frac{d}{dx}\left(\frac{y_2(x)}{y_1(x)}\right) = c$$

or,

$$y_2(x) = y_1(x)\int_{x_0}^{x} \frac{dt}{p(t)\left(y_1(t)\right)^2}.$$

Note:

- ▷ The previous theorem is not valid for periodic SL system, because in periodic system $W(y_1, y_2)(a)$ is not necessarily zero. See example 7.
- ▷ The thorem will hold for singular system with at least one homogeneous boundary condition.

Theorem 14. *Let λ_1 and $\lambda_2 (> \lambda_1)$ be the two eigenvalues of a regular SL system with corresponding eigenfunctions y_1 and y_2. There is a zero of y_2 between two successive zeros of y_1. In general, there are more zeros of y_2 than that of y_1.*

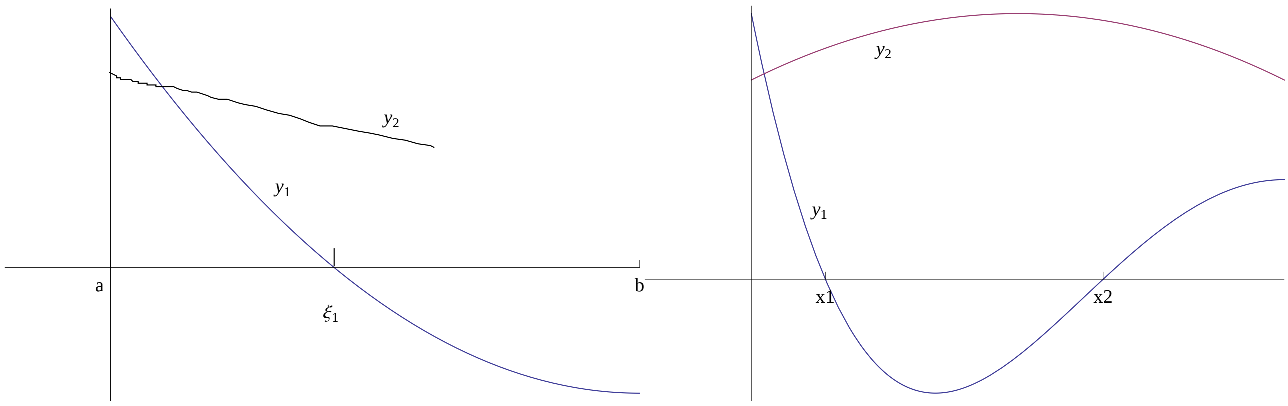

Figure 1: Case 1

Proof. Then

$$
\begin{aligned}
-\frac{d}{dx}\left[p(x)y_1'(x)\right] + qy_1(x) &= \lambda_1 y_1(x)w(x) \\
-\frac{d}{dx}\left[p(x)y_2'(x)\right] + qy_2(x) &= \lambda_2 y_2(x)w(x).
\end{aligned}
$$

Multiplying the first equation by y_2 and the second by y_2 and subtract. Then

$$
\frac{d}{dx}\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right] = \left(\lambda_1 - \lambda_2\right) y_1(x)y_2(x)w(x).
$$

If ξ_1 and ξ_2 are two points in $[a, b]$,

$$
\left[p(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right)\right]_{\xi_1}^{\xi_2} = \left(\lambda_1 - \lambda_2\right)\int_{\xi_1}^{\xi_2} y_1(x)y_2(x)w(x)dx \tag{5}
$$

Case 1: Now, consider a situation as shown in the fig . Without loss of genrality, let $y_1(a) > 0$. Let ζ be the first root of y_1 from a. Then $y_1(\zeta) = 0$ and $y_1'(\zeta) < 0$. Let $\xi_1 = a$ and $\xi_2 = \zeta$ in the equation 5:

$$
-p(\zeta)y_1'(\zeta)y_2(\zeta) = \left(\lambda_1 - \lambda_2\right)\int_{\xi_1}^{\xi_2} y_1(x)y_2(x)w(x)dx
$$

Now if we assume that $y_2(x) > 0$ for $x \in [a, \zeta]$, that is there is no zero of y_2 between a and ζ, the RHS of previous equation is negative, whereas the LHS is positive. This is a contradiction, which implies that y_2 must have a zero between a and ζ.

Case 2: Consider the second situation as shown in the fig. Let ξ_1 and ξ_2 be two successive zeroes of y_1, that is $y_1(\xi_1) = y_1(\xi_2) = 0$ and $y_1'(\xi_1) < 0$ and $y_1'(\xi_2) > 0$. Then the equation 5 becomes

$$
-p(\xi_2)y_1'(\xi_2)y_2(\xi_2) + p(\xi_1)y_1'(\xi_1)y_2(\xi_1) = \left(\lambda_1 - \lambda_2\right)\int_{\xi_1}^{\xi_2} y_1(x)y_2(x)w(x)dx
$$

Then if we assume that $y_2(x) > 0$ for $x \in [\xi_1, \xi_2]$, that is there is no zero of y_2 between ξ_1 and ξ_2, then the RHS is positive and the LHS is negative. Thus y_2 must have a zero between ξ_1 and ξ_2. □

2.3 Completeness of the set of eigenfunctions of SL System

In previous sections, we have defined the Hilbert space $\mathcal{H}$ as a subspace of $\mathcal{L}^2\left([a, b], w(x), dx\right)$ with functions satisfying the boundary conditions of a SL system defined on $[a, b]$. Claim is that the set of eigenfunctions of SL system forms a complete orthogonal basis of $\mathcal{H}$.

Let $\{y_n | n = 1, 2, \ldots\}$ be the set of normalized eigenfunctions of the SL system. If f is a function in $\mathcal{H}$, then

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} c_k y_k \right\| = 0$$

where,

$$c_k = \int_a^b \overline{y_k(x)} f(x) w(x) dx.$$

References:

1. *Linear Mathematics in Infinite Dimensions* by U H Gerlach,
   http://www.math.ohio-state.edu/∼gerlach/math/BVtypset/BVtypset.html