(b) How many subsets of X contain x_2 and x_3 but do not contain x_6?

Class Problems

Problem 15.5.
A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters—letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
$$\mathcal{D} = \{0, 1, 2, \ldots, 9\}$$

using unions ($\cup$) and set products ($\times$).

(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Problem 15.6. **(a)** How many of the billion numbers in the integer interval $[1..10^9]$ contain the digit 1? (*Hint:* How many don't?)

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected, and 15-bit strings with exactly 6 ones.

Problem 15.7.

(a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n. \tag{15.8}$$

That is

$$\mathcal{S}_{n,k} ::= \{(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k \mid (15.8) \text{ is true}\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

(b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \{(y_1, y_2, \ldots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \cdots \leq y_k \leq n\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Problem 15.8.
An n-vertex *numbered tree* is a tree whose vertex set is $[1..n]$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers in $[1..n]$ obtained by the following recursive process:[6]

> If there are more than two vertices left, write down the *father* of the largest leaf, delete this *leaf*, and continue this process on the resulting smaller tree. If there are only two vertices left, then stop—the code is complete.

For example, the codes of a couple of numbered trees are shown in the Figure 15.7.

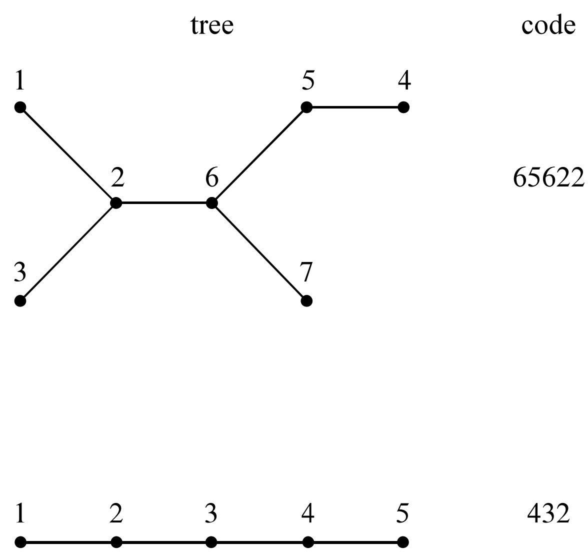

Figure 15.7

[6]The necessarily unique node adjacent to a leaf is called its *father*.

(a) Describe a procedure for reconstructing a numbered tree from its code.

(b) Conclude there is a bijection between the n-vertex numbered trees and sequences $(n-2)$ integers in $[1..n]$. State how many n-vertex numbered trees there are.

Problem 15.9.
Let X and Y be finite sets.

(a) How many binary relations from X to Y are there?

(b) Define a bijection between the set $[X \to Y]$ of all total functions from X to Y and the set $Y^{|X|}$. (Recall Y^n is the Cartesian product of Y with itself n times.) Based on that, what is $|[X \to Y]|$?

(c) Using the previous part, how many *functions*, not necessarily total, are there from X to Y? How does the fraction of functions vs. total functions grow as the size of X grows? Is it $O(1)$, $O(|X|)$, $O(2^{|X|})$,...?

(d) Show a bijection between the powerset $\text{pow}(X)$ and the set $[X \to \{0, 1\}]$ of 0-1-valued total functions on X.

(e) Let X be a set of size n and B_X be the set of all bijections from X to X. Describe a bijection from B_X to the set of permutations of X.[7] This implies that there are how many bijections from X to X?

Problems for Section 15.4

Class Problems

Problem 15.10.
Use induction to prove that there are 2^n subsets of an n-element set (Theorem 4.5.5).

[7] A sequence in which all the elements of a set X appear exactly once is called a *permutation* of X (see Section 15.3.3).

Homework Problems

Problem 15.11.
Fermat's Little Theorem 9.10.8[8] asserts that

$$a^p \equiv a \pmod{p} \tag{15.9}$$

for all primes p and nonnegative integers a. This is immediate for $a = 0, 1$ so we assume that $a \geq 2$.

This problem offers a proof of (15.9) by counting strings over a fixed alphabet with a characters.

(a) How many length-k strings are there over an a-character alphabet?

(b) How many of these strings use more than one character?

Let z be a length-k string. The *length-n rotation* of z is the string yx, where $z = xy$ and the length $|x|$ of x is $\text{rem}(n, k)$.

(c) Verify that if u is the length-n rotation of z, and v is the length-m rotation of u, then v is the length-$(n + m)$ rotation of z.

(d) Let $\approx$ be the "is a rotation of" relation on strings. That is,

$$v \approx z \quad \text{IFF} \quad \exists n \in \mathbb{N}. [v \text{ is a length-}n \text{ rotation of } z].$$

Prove that $\approx$ is an equivalence relation.

(e) Prove that if $xy = yx$ then x and y each consist of repetitions of some string u. That is, if $xy = yx$, then $x, y \in u^*$ for some string u.

Hint: By induction on the length $|xy|$ of xy.

(f) Conclude that if p is prime and z is a length-p string containing at least two different characters, then z is equivalent under $\approx$ to exactly p strings (counting itself).

(g) Conclude from parts (a) and (f) that $p \mid (a^p - a)$, which proves Fermat's Little Theorem (15.9).

[8] This Theorem is usually stated as

$$a^{p-1} \equiv 1 \pmod{p},$$

for all primes p and integers a not divisible by p. This follows immediately from (15.9) by canceling a.

Problems for Section 15.5

Practice Problems

Problem 15.12.
Eight students—Anna, Brian, Caine,...—are to be seated around a circular table in a circular room. Two seatings are regarded as defining the same *arrangement* if each student has the same student on his or her right in both seatings: it does not matter which way they face. We'll be interested in counting how many arrangements there are of these 8 students, given some restrictions.

(a) As a start, how many different arrangements of these 8 students around the table are there without any restrictions?

(b) How many arrangements of these 8 students are there with Anna sitting next to Brian?

(c) How many arrangements are there with if Brian sitting next to both Anna AND Caine?

(d) How many arrangements are there with Brian sitting next to Anna *OR* Caine?

Problem 15.13.
How many different ways are there to select an unordered bundle of three dozen colored roses if red, yellow, pink, white, purple and orange roses are available? Please explain your answer (you may leave it un-simplified).

Problem 15.14.
Suppose n books are lined up on a shelf. The number of selections of m of the books so that selected books are separated by at least three unselected books is the same as the number of *all* length k binary strings with exactly m ones.

(a) What is the value of k?

(b) Describe a bijection between between the set of all length k binary strings with exactly m ones and such book selections.

Problem 15.15.
Six women and nine men are on the faculty of a school's EECS department. The

individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?

Class Problems

Problem 15.16.
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

(c) How many group assignments are possible?

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Problem 15.17.
A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22,369,621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ can't be an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Problem 15.18.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{ccc} (a,b) & (a,c) & (a,d) \\ (b,a) & (b,c) & (b,d) \\ (c,a) & (c,b) & (c,d) \\ (d,a) & (d,b) & (d,c) \end{array}$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Problem 15.19. **(a)** There are 30 books arranged in a row on a shelf. In how many ways can eight of these books be selected so that there are at least two unselected books between any two selected books?

(b) How many nonnegative integer solutions are there for the following equality?

$$x_1 + x_2 + \cdots + x_m = k. \tag{15.10}$$

(c) How many nonnegative integer solutions are there for the following inequality?

$$x_1 + x_2 + \cdots + x_m \leq k. \tag{15.11}$$

(d) How many length m weakly increasing sequences of nonnegative integers $\leq k$ are there?