State a similar propositional equivalence that would justify the key step in a proof for the following set equality organized as a chain of iff's:

$$\overline{A \cap B \cap C} = \overline{A} \cup (\overline{B} - \overline{A}) \cup \overline{C}.$$

(You are *not* being asked to write out an iff-proof of the equality or to write out a proof of the propositional equivalence. Just state the equivalence.)

**Problem 4.13.**
The set equation

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

follows from a certain equivalence between propositional formulas.

**(a)** What is the equivalence?

**(b)** Show how to derive the equation from this equivalence.

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## Problems for Section 4.2

### Homework Problems

**Problem 4.14.**
Prove that for any sets $A$, $B$, $C$ and $D$, if the Cartesian products $A \times B$ and $C \times D$ are disjoint, then either $A$ and $C$ are disjoint or $B$ and $D$ are disjoint.

**Problem 4.15.** **(a)** Give a simple example where the following result fails, and briefly explain why:

**False Theorem.** *For sets $A$, $B$, $C$ and $D$, let*

$$L ::= (A \cup B) \times (C \cup D),$$
$$R ::= (A \times C) \cup (B \times D).$$

*Then $L = R$.*

**(b)** Identify the mistake in the following proof of the False Theorem.

*Bogus proof.* Since $L$ and $R$ are both sets of pairs, it's sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all $x, y$.

The proof will be a chain of iff implications:

**Problem 4.18.**

For a binary relation $R : A \to B$, some properties of $R$ can be determined from just the arrows of $R$, that is, from graph($R$), and others require knowing if there are elements in the domain $A$ or the codomain $B$ that don't show up in graph($R$). For each of the following possible properties of $R$, indicate whether it is always determined by

1. graph($R$) alone,
2. graph($R$) and $A$ alone,
3. graph($R$) and $B$ alone,
4. all three parts of $R$.

Properties:

**(a)** surjective

**(b)** injective

**(c)** total

**(d)** function

**(e)** bijection

**Problem 4.19.**

For each of the following real-valued functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

**(a)** $x \to x + 2$

**(b)** $x \to 2x$

**(c)** $x \to x^2$

**(d)** $x \to x^3$

**(e)** $x \to \sin x$

**(f)** $x \to x \sin x$

**(g)** $x \to e^x$

**(b)** Show there is a total injection $f$ and a bijection, $g$, such that $g \circ f$ is not a bijection.

**Problem 4.26.**
Let $A$, $B$ and $C$ be nonempty sets, and let $f : B \to C$ and $g : A \to B$ be functions. Let $h ::= f \circ g$ be the composition function of $f$ and $g$, namely, the function with domain $A$ and codomain $C$ such that $h(x) = f(g(x))$.

**(a)** Prove that if $h$ is surjective and $f$ is total and injective, then $g$ must be surjective.

*Hint:* contradiction.

**(b)** Suppose that $h$ is injective and $f$ is total. Prove that $g$ must be injective and provide a counterexample showing how this claim could fail if $f$ was *not* total.

**Problem 4.27.**
Let $A$, $B$ and $C$ be sets, and let $f : B \to C$ and $g : A \to B$ be functions. Let $h : A \to C$ be the composition $f \circ g$; that is, $h(x) ::= f(g(x))$ for $x \in A$. Prove or disprove the following claims:

**(a)** If $h$ is surjective, then $f$ must be surjective.

**(b)** If $h$ is surjective, then $g$ must be surjective.

**(c)** If $h$ is injective, then $f$ must be injective.

**(d)** If $h$ is injective and $f$ is total, then $g$ must be injective.

**Problem 4.28.** **(a)**

Let $R : D \to D$ be a binary relation on a set $D$. Let $x, y$ be variables ranging over $D$. Indicate the expressions below whose meaning is that $R$ is an *injective relation* [$\leq$ 1 in]. Remember that $R(x) ::= \{y \mid x \; R \; y\}$, and $R$ is not necessarily a function or a total relation.

(i) $R(x) \cap R(y) = \emptyset$

(ii) $R(x) = R(y)$ IMPLIES $x = y$

(iii) $R(x) \cap R(y) = \emptyset$ IMPLIES $x \neq y$

(iv) $x \neq y$ IMPLIES $R(x) \neq R(y)$

**(e)** The set of pages that have at least one incoming or outgoing link

**(f)** The relation that relates word $w$ and page $p$ iff $w$ appears on a page that links to $p$

**(g)** The relation that relates word $w$ and endorser $e$ iff $w$ appears on a page that links to a page that $e$ recommends

**(h)** The relation that relates pages $p_1$ and $p_2$ iff $p_2$ can be reached from $p_1$ by following a sequence of exactly 3 links

## Exam Problems

**Problem 4.30.**
Let $L_n$ be the length-$n$ sequences of digits [0..9] whose sum of digits is less than $9n/2$, and let $G_n$ be the length-$n$ sequences of digits [0..9] whose sum of digits is greater than $9n/2$. For example, $\texttt{2134} \in L_4$ and $\texttt{9786} \in G_4$.

Describe a simple bijection between $L_n$ and $G_n$. Prove carefully that your mapping is a bijection by verifying that it is a function, total, injective, and surjective.

**Problem 4.31.**
Let $A$ be the set containing the five sets: $\{a\}, \{b, c\}, \{b, d\}, \{a, e\}, \{e, f\}$, and let $B$ be the set containing the three sets: $\{a, b\}, \{b, c, d\}, \{e, f\}$. Let $R$ be the "is subset of" binary relation from $A$ to $B$ defined by the rule:

$$X \; R \; Y \quad \text{IFF} \quad X \subseteq Y.$$

**(a)** Fill in the arrows so the following figure describes the graph of the relation, $R$:

**Problem 4.34.**
Prove that if relation $R : A \to B$ is a total injection, [$\geq 1$ out], [$\leq 1$ in], then

$$R^{-1} \circ R = \text{Id}_A,$$

where $\text{Id}_A$ is the identity function on $A$.
(A simple argument in terms of "arrows" will do the job.)

**Problem 4.35.**
Let $R : A \to B$ be a binary relation.

**(a)** Prove that $R$ is a function iff $R \circ R^{-1} \subseteq \text{Id}_B$.

Write similar containment formulas involving $R^{-1} \circ R$, $R \circ R^{-1}$, $\text{Id}_a$, $\text{Id}_B$ equivalent to the assertion that $R$ has each of the following properties. No proof is required.

**(b)** total.

**(c)** a surjection.

**(d)** a injection.

**Problem 4.36.**
Let $R : A \to B$ and $S : B \to C$ be binary relations such that $S \circ R$ is a bijection and $|A| = 2$.

Give an example of such $R, S$ where neither $R$ nor $S$ is a function. Indicate exactly which properties—total, surjection, function, and injection—your examples of $R$ and $S$ have.

*Hint:* Let $|B| = 4$.

**Problem 4.37.**
The set $\{1, 2, 3\}^\omega$ consists of the **infinite** sequences of the digits 1,2, and 3, and likewise $\{4, 5\}^\omega$ is the set of infinite sequences of the digits 4,5. For example

$$\begin{aligned} &\texttt{123123123...} && \in \{1, 2, 3\}^\omega, \\ &\texttt{222222222222...} && \in \{1, 2, 3\}^\omega, \\ &\texttt{4554445554444...} && \in \{4, 5\}^\omega. \end{aligned}$$

**(a)** Give an example of a total injective function

$$f : \{1, 2, 3\}^\omega \to \{4, 5\}^\omega.$$

**(b)** Give an example of a bijection $g : (\{1, 2, 3\}^\omega \times \{1, 2, 3\}^\omega) \to \{1, 2, 3\}^\omega$.