whatever familiar facts about integers and primes you need, but explicitly state such facts.

Problems for Section 1.7

Practice Problems

Problem 1.7.
Prove by cases that

$$\max(r, s) + \min(r, s) = r + s \tag{*}$$

for all real numbers r, s.

Class Problems

Problem 1.8.
If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Problem 1.9.
Prove by cases that

$$|r + s| \leq |r| + |s| \tag{1}$$

for all real numbers r, s.[9]

Homework Problems

Problem 1.10. **(a)** Suppose that

$$a + b + c = d,$$

where a, b, c, d are nonnegative integers.

Let P be the assertion that d is even. Let W be the assertion that exactly one among a, b, c are even, and let T be the assertion that all three are even.

Prove by cases that

$$P \text{ IFF } [W \text{ OR } T].$$

[9] The *absolute value* $|r|$ of r equals whichever of r or $-r$ is not negative.

(b) Now suppose that

$$w^2 + x^2 + y^2 = z^2,$$

where w, x, y, z are nonnegative integers. Let P be the assertion that z is even, and let R be the assertion that all three of w, x, y are even. Prove by cases that

$$P \text{ IFF } R.$$

Hint: An odd number equals $2m + 1$ for some integer m, so its square equals $4(m^2 + m) + 1$.

Exam Problems

Problem 1.11.
Prove that there is an irrational number a such that $a^{\sqrt{3}}$ is rational.
Hint: Consider $\sqrt[3]{2}^{\sqrt{3}}$ and argue by cases.

Problems for Section 1.8

Practice Problems

Problem 1.12.
Prove that for any $n > 0$, if a^n is even, then a is even.
Hint: Contradiction.

Problem 1.13.
Prove that if $a \cdot b = n$, then either a or b must be $\leq \sqrt{n}$, where a, b, and n are nonnegative real numbers. *Hint:* by contradiction, Section 1.8.

Problem 1.14.
Let n be a nonnegative integer.

(a) Explain why if n^2 is even—that is, a multiple of 2—then n is even.

(b) Explain why if n^2 is a multiple of 3, then n must be a multiple of 3.

Problem 1.15.
Give an example of two distinct positive integers m, n such that n^2 is a multiple of

m, but n is not a multiple of m. How about having m be less than n?

Class Problems

Problem 1.16.
How far can you generalize the proof of Theorem 1.8.1 that $\sqrt{2}$ is irrational? For example, how about $\sqrt{3}$?

Problem 1.17.
Prove that $\log_4 6$ is irrational.

Problem 1.18.
Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.
Hint: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

Problem 1.19.
Here is a generalization of Problem 1.16 that you may not have thought of:

Lemma. *Let the coefficients of the polynomial*

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-1} x^{m-1} + x^m$$

be integers. Then any real root of the polynomial is either integral or irrational.

(a) Explain why the Lemma immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an mth power of some integer.

(b) Carefully prove the Lemma.

You may find it helpful to appeal to:
Fact. If a prime p is a factor of some power of an integer, then it is a factor of that integer.

You may assume this Fact without writing down its proof, but see if you can explain why it is true.

Exam Problems

Problem 1.20.
Prove that $\log_9 12$ is irrational.

Problem 2.6.
You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

> **Property** m: For any nonnegative integer less than 2^{m+1}, there is a selection of envelopes whose contents add up to *exactly* that number of dollars.

Use the Well Ordering Principle (WOP) to prove that Property m holds for all nonnegative integers m.

Hint: Consider two cases: first, when the target number of dollars is less than 2^m and second, when the target is at least 2^m.

Homework Problems

Problem 2.7.
Use the Well Ordering Principle to prove that any integer greater than or equal to 8 can be represented as the sum of nonnegative integer multiples of 3 and 5.

Problem 2.8.
Use the Well Ordering Principle to prove that any integer greater than or equal to 50 can be represented as the sum of nonnegative integer multiples of 7, 11, and 13.

Problem 2.9.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation were first discovered in 1986. So Euler guessed wrong, but it took more than two centuries to demonstrate his mistake.

Now let's consider Lehman's equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{2.5}$$

Prove that Lehman's equation (2.5) really does not have any positive integer solutions.

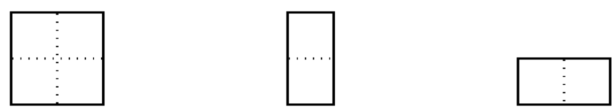
Hint: Consider the minimum value of a among all possible solutions to (2.5).

Problem 2.10.
Use the Well Ordering Principle to prove that

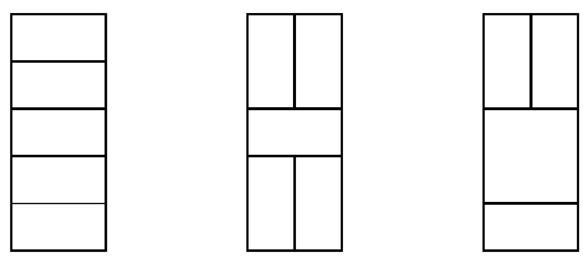
$$n \leq 3^{n/3} \qquad (*)$$

for every nonnegative integer n.
Hint: Verify (*) for $n \leq 4$ by explicit calculation.

Problem 2.11.
A *winning configuration* in the game of Mini-Tetris is a complete tiling of a $2 \times n$ board using only the three shapes shown below:



For example, here are several possible winning configurations on a 2×5 board:



(a) Let T_n denote the number of different winning configurations on a $2 \times n$ board. Determine the values of T_1, T_2 and T_3.

(b) Express T_n in terms of T_{n-1} and T_{n-2} for $n > 2$.

(c) Use the Well Ordering Principle to prove that the number of winning configurations on a $2 \times n$ Mini-Tetris board is:[3]

$$T_n = \frac{2^{n+1} + (-1)^n}{3} \qquad (*)$$

Problem 2.12.
Mini-Tetris is a game whose objective is to provide a complete "tiling" of a $n \times 2$ board—that is, a board consisting of two length-n columns—using tiles of specified shapes. In this problem we consider the following set of five tiles:

[3] A good question is how someone came up with equation (*) in the first place. A simple Well Ordering proof gives no hint about this, but it should be absolutely convincing anyway.

Proof. We use induction. Let $P(n)$ be the proposition that $2 + 3 + 4 + \cdots + n = n(n+1)/2$.

Base case: $P(0)$ is true, since both sides of the equation are equal to zero. (Recall that a sum with no terms is zero.)

Inductive step: Now we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 0$. So suppose that $P(n)$ is true; that is, $2 + 3 + 4 + \cdots + n = n(n+1)/2$. Then we can reason as follows:

$$\begin{aligned} 2 + 3 + 4 + \cdots + n + (n+1) &= [2 + 3 + 4 + \cdots + n] + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

Above, we group some terms, use the assumption $P(n)$, and then simplify. This shows that $P(n)$ implies $P(n+1)$. By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. ■

Where exactly is the error in this proof?

Homework Problems

Problem 5.8.

The Fibonacci numbers $F(n)$ are described in Section 5.2.2.

Prove by induction that for all $n \geq 1$,

$$F(n-1) \cdot F(n+1) - F(n)^2 = (-1)^n. \tag{5.8}$$

Problem 5.9.

For any binary string α let $\text{num}(\alpha)$ be the nonnegative integer it represents in binary notation (possibly with leading zeroes). For example, $\text{num}(\texttt{10}) = 2$, and $\text{num}(\texttt{0101}) = 5$.

An $n+1$-*bit adder* adds two $n+1$-bit binary numbers. More precisely, an $n+1$-bit adder takes two length $n+1$ binary strings

$$\begin{aligned} \alpha_n &::= a_n \ldots a_1 a_0, \\ \beta_n &::= b_n \ldots b_1 b_0, \end{aligned}$$

and a binary digit c_0 as inputs, and produces a length-$(n+1)$ binary string

$$\sigma_n ::= s_n \ldots s_1 s_0,$$

and a binary digit c_{n+1} as outputs, and satisfies the specification:

$$\text{num}(\alpha_n) + \text{num}(\beta_n) + c_0 = 2^{n+1}c_{n+1} + \text{num}(\sigma_n). \tag{5.9}$$

There is a straighforward way to implement an $n+1$-bit adder as a digital circuit: an $n+1$-bit *ripple-carry circuit* has $1 + 2(n+1)$ binary inputs

$$a_n, \ldots, a_1, a_0, b_n, \ldots, b_1, b_0, c_0,$$

and $n+2$ binary outputs,

$$c_{n+1}, s_n, \ldots, s_1, s_0.$$

As in Problem 3.6, the ripple-carry circuit is specified by the following formulas:

$$s_i ::= a_i \text{ XOR } b_i \text{ XOR } c_i \tag{5.10}$$

$$c_{i+1} ::= (a_i \text{ AND } b_i) \text{ OR } (a_i \text{ AND } c_i) \text{ OR } (b_i \text{ AND } c_i) \tag{5.11}$$

for $0 \le i \le n$, where we follow the convention that 1 corresponds to **T** and 0 corresponds to **F**.

(a) Verify that definitions (5.10) and (5.11) imply that

$$a_n + b_n + c_n = 2c_{n+1} + s_n. \tag{5.12}$$

for all $n \in \mathbb{N}$.

(b) Prove by induction on n that an $n+1$-bit ripple-carry circuit really is an $n+1$-bit adder, that is, its outputs satisfy (5.9).

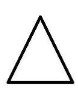
Hint: You may assume that, by definition of binary representation of integers,

$$\text{num}(\alpha_{n+1}) = a_{n+1}2^{n+1} + \text{num}(\alpha_n). \tag{5.13}$$

Problem 5.10.

Divided Equilateral Triangles[5] (DETs) can be built up as follows:

- A single equilateral triangle counts as a DET whose only unit subtriangle is itself.

- If $T ::=$  is a DET, then the equilateral triangle T' built out of four copies of T as shown in in Figure 5.9 is also a DET, and the unit subtriangles of T' are exactly the unit subtriangles of each of the copies of T.

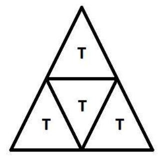


Figure 5.9 DET T' from Four Copies of DET T

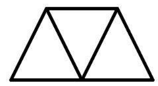


Figure 5.10 Trapezoid from Three Triangles

From here on "subtriangle" will mean unit subtriangle.

(a) Define the *length* of a DET to be the number of subtriangles with an edge on its base. Prove **by induction on length** that the total number of subtriangles of a DET is the square of its length.

(b) Show that a DET with one of its corner subtriangles removed can be tiled with trapezoids built out of three subtriangles as in Figure 5.10.

Problem 5.11.
The Math for Computer Science mascot, Theory Hippotamus, made a startling discovery while playing with his prized collection of unit squares over the weekend. Here is what happened.

First, Theory Hippotamus put his favorite unit square down on the floor as in Figure 5.11 (a). He noted that the length of the periphery of the resulting shape was 4, an even number. Next, he put a second unit square down next to the first so that the two squares shared an edge as in Figure 5.11 (b). He noticed that the length of the periphery of the resulting shape was now 6, which is also an even number. (The periphery of each shape in the figure is indicated by a thicker line.) Theory Hippotamus continued to place squares so that each new square shared an edge with at least one previously-placed square and no squares overlapped. Eventually, he arrived at the shape in Figure 5.11 (c). He realized that the length of the periphery of this shape was 36, which is again an even number.

[5] Adapted from [49].