CS 514, Mathematics for Computer Science
End-semester Exam, Monsoon 2019
Department of Computer Science and Engineering
IIT Guwahati
Time: Three hours

Important

1. No questions about the paper will be entertained during the exam.
2. You must answer each question in the space provided for that question in the **answer sheet**. Answers appearing outside the space provided will not be considered.
3. Keep your rough work separate from your answers. A supplementary sheet is being provided for rough work. **Do not attach your rough work to the answer sheet.**
4. This exam has 9 questions over 7 pages, with a total of 90 marks.
5. **No credit will be given for complicated answers, even if correct, when a simple answer exists.**
6. **Write your roll number at the top of every page in the answer sheet.**

1. A simple graph G is said to have *width* w if its vertices can be arranged in a sequence such that each vertex is adjacent to at most w vertices that precede it in the sequence. Prove that every graph with *width* w is $(w+1)$-colourable.

(10)

Solution: We prove the result by induction on the number n of vertices of the graph.

Basis: For $n = 1$, the single vertex graph is clearly $(w+1)$-colourable since just one colour is needed.

Induction Step: Suppose the result is true for any graph with $n = k$ vertices. Take a graph G with $k+1$ vertices of width w. Then the vertices can be arranged in a sequence $v_1, v_2, \ldots, v_k, v_{k+1}$ such that each vertex is adjacent to at most w vertices that precede it in the sequence. By the induction hypothesis, the graph G' obtained from G by dropping the last vertex v_{k+1} along with the edges incident on it is $(w+1)$-colourable, since it is of width w. Now consider the vertex v_{k+1} in G. Since it is adjacent to at most w vertices and $w+1$ colours are available, it can be coloured as well. Hence G is $(w+1)$-colourable.

2. The n-dimensional hypercube H_n is a graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly one bit. For example, in H_3, vertices 111 and 011 are adjacent because they differ only in the first

bit, while vertices 101 and 011 are not adjacent because they differ at both the first and second bits.

(a) Prove that it is impossible to find two spanning trees of H_3 that do not share some edge.

(5)

Solution: In the hypercube H_3, the number of nodes is 8 and the number of edges is 12. The number of edges in any spanning tree of H_3 is $8 - 1 = 7$. Now $7 + 7 = 14 > 12$. Therefore, any two spanning trees will share at least one edge.

(b) What is the highest number k such that H_3 is k-connected.

Solution: In H_3, the highest number k such that H_3 is k-connected is 3. This is because removing three edges adjacent to any node disconnects the graph, but removing any two edges still leaves a connected graph.

(5)

3. A simple graph G is *triangle-free* when it has no cycle of length three.

(a) Prove that for any connected triangle-free planar graph with $v > 2$ vertices and e edges, $e \leq 2v - 4$.

(5)

Solution: Since there are no 3-cycles, each face has at least 4 edges. Since each edge is counted twice in the sum of the lengths of the face bounds, we have $4f \leq 2e$, *i.e.*, $2f \leq e$, where f is the number of faces. By Euler's formula, $e - v + 2 = f$, so $2(e - v + 2) = 2f \leq e$ which gives $e \leq 2v - 4$.

(b) Show that any connected triangle-free planar graph has at least one vertex of degree three or less.

Solution: We have $e \leq 2v - 4$ from the result proved in (a). Now since the the sum of the degrees is twice the number of edges, and

$$2e \leq 4v - 8 < 4v$$

the sum is strictly less than $4v$. Therefore, some vertex must have degree strictly less than 4.

(5)

4. Let A_i be the set of length-n binary strings in which 011 occurs starting at the i^{th} position where the positions are numbered from 0 to $n-1$. Note that A_i is the empty set for $i > n-3$. For any $i \neq j$, the intersection $A_i \cap A_j$ is either empty or of size s. Write a formula for s in terms of n. Justify your answer. (10)

Solution: We are given that

$$A_i = \{b_0 b_2 \ldots b_{i-1} 011 b_{i+3} \ldots b_{n-1} \mid b_k \in \{0,1\} \text{ for } 1 \leq k \leq n\}.$$

When $i \neq j$, $A_i \cap A_j = \emptyset$ if $|i-j| \in \{1,2\}$ or any of i or j is greater than $n-3$. Otherwise,

$$s = |A_i \cap A_j| = 2^{n-6}$$

since out of the n positions in the intersection, six positions have been fixed, and the other $n-6$ positions can have either 0 or 1.

5. A robot on a point in the 3-D integer lattice can move a unit distance in one positive direction at a time. That is, from position (x, y, z) it can move to either $(x+1, y, z)$, $(x, y+1, z)$ or $(x, y, z+1)$. For any two points P and Q in the lattice, let $n(P, Q)$ denote the number of distinct paths the robot can follow to go from P to Q. Let

$$A = (0, 10, 20), B = (30, 50, 70), C = (80, 90, 100).$$

Answer the following questions with proper justification.

(a) Express $n(A, B)$ as a numerical formula that is a quotient $\frac{a}{b}$ where a and b are products of factorials. Justify your answer. (5)

Solution: To go from $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$ where $x_1 < y_1, x_2 < y_2$ and $z_1 < z_2$, the robot needs to take $\Delta_x = (x_2 - x_1)$ steps along the X-axis, $\Delta_y = (y_2 - y_1)$ steps along the Y-axis and $\Delta_z = (z_2 - z_1)$ steps along the Z-axis. The number of paths $n(P, Q)$ is therefore the set of words (sequences) with three letters (corresponding to the three axes) where there are Δ_x occurrences of the first letter, Δ_y occurrences of the second letter and Δ_z occurrences of the third letter. This is the multinomial coefficient

$$\binom{\Delta_x + \Delta_y + \Delta_z}{\Delta_x, \Delta_y, \Delta_z} = \frac{(\Delta_x + \Delta_y + \Delta_z)!}{\Delta_x! \Delta_y! \Delta_z!}.$$

Therefore, the answer is
$$\frac{120!}{30!40!50!}.$$

(b) How many paths from A to C go through B?

(5)

Solution: Any path from A to C through B is a path from A to B followed by a path from B to C. Using the product rule, the answer is

$$n(A,B).n(B,c) = \frac{120!}{30!40!50!}.\frac{120!}{50!40!30!} = (\frac{120!}{50!40!30!})^2.$$

6. Define the *Double Fibonacci* numbers D_n recursively by the rules

$$D_0 = D_1 = 1$$
$$D_n = 2D_{n-1} + D_{n-2} \text{ for } n \geq 2$$

Express the generating function $D(x)$ for the Double Fibonacci numbers as a quotient of polynomials. You do *not* have to find a formula for $[x^n]D(x)$.

(10)

Solution: Let the generating function for the Double Fibonacci numbers be

$$D(x) = d_0 + d_1x + d_2x^2 + d_3x^3 + \dots$$

where $d_0 = d_1 = 1$ and $d_n = 2d_{n-1} + d_{n-2}$ for $n \geq 2$. Then we have the following equations:

$$D(x) = d_0 + \quad d_1x + d_2x^2 + d_3x^3 \quad + \dots d_nx^n + \dots \qquad (1)$$
$$2xD(x) = \quad 2d_0x + 2d_1x^2 + 2d_2x^3 \quad + \dots 2d_{n-1}x^n + \dots \qquad (2)$$
$$x^2D(x) = \quad d_0x^2 + d_1x^3 \quad + \dots d_{n-2}x^n + \dots \qquad (3)$$

Then by adding equations 2 and 3 and subtracting from equation 1 we have

$$(1 - 2x - x^2)D(x) = d_0 + (d_1 - 2d_0)x + 0 + 0 + 0 \dots = 1 - x \text{ since } d_0 = d_1 = 1.$$

Therefore,
$$D(x) = \frac{1-x}{1-2x-x^2}.$$

7. Answer the following questions with proper justification.

   (a) Suppose that you *randomly* permute the numbers $1, 2, \ldots, n$, that is, you select a permutation uniformly at random. Given numbers i and k in the range $1, \ldots, n$ what is the probability the number k ends up in the i^{th} position after the permutation? (5)

   Solution: There are $n!$ outcomes each of which has probability $\frac{1}{n!}$. For fixed k and i the number of permutations with k in the i^{th} position is $(n-1)!$ since there are $n-1$ remaining psoitions to be filled with $(n-1)$ numbers. Hence, by the sum rule, the required probability is $\frac{(n-1)!}{n!} = \frac{1}{n}$.

   (b) A fair coin is flipped n times. What is the probability that all the heads occur at the end of the sequence? (If no heads occur, then "all the heads are at the end of the sequence" is vacuously true.) (5)

   Solution: The probability that i heads occur at the end (and the rest are tails) is $(\frac{1}{2})^n$ since there is only one sequence among a total of 2^n sequences with the property. Therefore, the probability that all heads are at the end is

   $$\Sigma_{i=0}^{n}(\frac{1}{2})^n = \frac{n+1}{2^n}.$$

8. Two teams A and B participate in a two-out-of-three series, *i.e.*, they play until one team has won two games. Then that team is declared the overall winner and the series ends. Assume that Team A win each game with probability $\frac{3}{5}$, regardless of the outcomes of previous games. Answer the following questions with proper justification.

   (a) What is the probability that a total of 3 games are played? (5)

   Solution: The sample space along with the probabilities of individual outcomes are given by:

   $$\begin{array}{cccc} S = \{AA, & ABA, ABB, & BAA, BB, & BAB\} \\ \frac{9}{25} & \frac{18}{125}\ \frac{12}{125} & \frac{18}{125}\ \frac{4}{25} & \frac{12}{125} \end{array}$$

   where, for example, ABA is the outcome in which A wins the first game, B the second and A the third. The event for a total of 3 games being played is given by

$$E = \{ABA, ABB, BAA, BAB\}$$

and its probability is

$$Pr\{E\} = \frac{18 + 12 + 18 + 12}{125} = \frac{60}{125} = \frac{12}{25}.$$

(b) What is the probability that the winner of the series loses the first game?

Solution: The event for the winner of the series losing the first game is

$$F = \{ABB, BAA\}$$

and its probability is

$$Pr\{F\} = \frac{12 + 18}{125} = \frac{30}{125} = \frac{6}{25}.$$

(5)

9. Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the i^{th} component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The system will fail if at least one of its components fails.

(a) What is the probability that the system will fail within one year? The answer should be as simple as possible and not involve complicated expressions.

(5)

Solution: The probability of a single component *not* failing in a given year is $1 - p$. Therefore the probability of the system *not* failing in a year is $(1 - p)^n$, since there are n components, where we assume that the failures of different components are independent. Therefore, the probability that the system will fail within one year is $1 - (1 - p)^n$.

(b) What is the probability that the system will fail within k years, for $k \geq 1$?

(5)

Solution: From (a) above the probability of the system *not* failing in a given year is $(1-p)^n$. Therefore, the probability of the system *not* failing in the first k years is $r = (1-p)^{nk}$. Hence, the probability that the system will fail within k years is $(1-r) = 1-(1-p)^{nk}$.