Problems for Section 3.6

Practice Problems

Problem 3.24.
For each of the following propositions:

1. $\forall x\, \exists y.\ 2x - y = 0$
2. $\forall x\, \exists y.\ x - 2y = 0$
3. $\forall x.\ x < 10$ IMPLIES $(\forall y.\ y < x$ IMPLIES $y < 9)$
4. $\forall x\, \exists y.\ [y > x \wedge \exists z.\ y + z = 100]$

determine which propositions are true when the variables range over:

(a) the nonnegative integers.

(b) the integers.

(c) the real numbers.

Problem 3.25.
Let $Q(x, y)$ be the statement

"x has been a contestant on television show y."

The universe of discourse for x is the set of all students at your school and for y is the set of all quiz shows that have ever been on television.

Determine whether or not each of the following expressions is logically equivalent to the sentence:

"No student at your school has ever been a contestant on a television quiz show."

(a) $\forall x\, \forall y.$ NOT$(Q(x, y))$

(b) $\exists x\, \exists y.$ NOT$(Q(x, y))$

(c) NOT$(\forall x\, \forall y.\ Q(x, y))$

(d) NOT$(\exists x\, \exists y.\ Q(x, y))$

Problem 3.26.
Express each of the following statements using quantifiers, logical connectives, and/or the following predicates

$P(x)$**:** x is a monkey,

$Q(x)$**:** x is a 6.042 TA,

$R(x)$**:** x comes from the 23rd century,

$S(x)$**:** x likes to eat pizza,

where x ranges over all living things.

(a) No monkeys like to eat pizza.

(b) Nobody from the 23rd century dislikes eating pizza.

(c) All 6.042 TAs are monkeys.

(d) No 6.042 TA comes from the 23rd century.

(e) Does part (d) follow logically from parts (a), (b), (c)? If so, give a proof. If not, give a counterexample.

(f) Translate into English: $(\forall x)(R(x) \vee S(x) \longrightarrow Q(x))$.

(g) Translate into English:

$$[\exists x.\ R(x) \text{ AND NOT}(Q(x))] \text{ IMPLIES } \forall x.\ (P(x) \text{ IMPLIES } S(x)).$$

Problem 3.27.
Find a counter-model showing the following is not valid.

$$\exists x. P(x) \text{ IMPLIES } \forall x. P(x)$$

(Just define your counter-model. You do not need to verify that it is correct.)

Problem 3.28.
Find a counter-model showing the following is not valid.

$$[\exists x.\ P(x) \text{ AND } \exists x. Q(x)] \text{ IMPLIES } \exists x.[P(x) \text{ AND } Q(x)]$$

(Just define your counter-model. You do not need to verify that it is correct.)

Problem 3.29.
Which of the following are *valid*? For those that are not valid, desribe a counter-model.

(a) $\exists x \exists y.\ P(x, y)$ IMPLIES $\exists y \exists x.\ P(x, y)$

(b) $\forall x \exists y.\ Q(x, y)$ IMPLIES $\exists y \forall x.\ Q(x, y)$

(c) $\exists x \forall y.\ R(x, y)$ IMPLIES $\forall y \exists x.\ R(x, y)$

(d) NOT$(\exists x\ S(x))$ IFF $\forall x$ NOT$(S(x))$

Problem 3.30. **(a)** Verify that the propositional formula

$$(P \text{ IMPLIES } Q) \text{ OR } (Q \text{ IMPLIES } P)$$

is valid.

(b) The valid formula of part (a) leads to sound proof method: to prove that an implication is true, just prove that its converse is false.[5] For example, from elementary calculus we know that the assertion

If a function is continuous, then it is differentiable

is false. This allows us to reach at the correct conclusion that its converse,

If a function is differentiable, then it is continuous

is true, as indeed it is.

But wait a minute! The implication

If a function is differentiable, then it is not continuous

is completely false. So we could conclude that its converse

If a function is not continuous, then it is differentiable,

should be true, but in fact the converse is also completely false.

So something has gone wrong here. Explain what.

[5] This problem was stimulated by the discussion of the fallacy in [4].