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Optimal Mistuning for Improved Stability of Vehicular Platoons
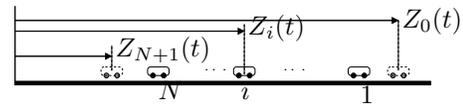
Prabir Barooah

Prashant G. Mehta

(a) A platoon with fictitious lead and follow vehicles.

(b) Same platoon in y coordinates.

Fig. 1. A platoon with N vehicles moving in one dimension.

Abstract— We consider a decentralized bidirectional control of a platoon of N identical vehicles moving in a straight line. Such problems are known to suffer from poor stability margin and sensitivity to disturbance as the number of vehicles gets large. In this paper, we present a novel control design methodology for optimization of the stability margin using distributed control. The methodology employs a variational formulation for minimization of the least stable eigenvalue of a partial differential equation (PDE) approximation of the platoon dynamics. We show that the eigenvalue optimization based control has better closed-loop stability margin and sensitivity to disturbance than a symmetric architecture where the same control law is used by each vehicle. All the conclusions drawn from analysis of the PDE model are corroborated via numerical calculations of the discrete platoon model.
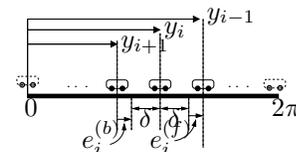
I. INTRODUCTION

We consider the problem of control law design for distributed control of a one-dimensional platoon of N identical vehicles. The control objective is that every vehicle move at a constant pre-specified velocity V_d with an inter-vehicular spacing of Δ. The control action at an individual vehicle depends upon its own velocity and the relative position errors between itself and its predecessor and its follower vehicles (see Figure 1(a)).

Decentralized control of large vehicular platoons suffers from several challenges. First, the least stable closed-loop eigenvalue approaches zero as the number of vehicles, N, increases [1], [2]. This progressive loss of closed-loop damping causes the closed loop performance of the platoon to become arbitrarily sluggish as the number of vehicles gets large. Second, with decentralized control the sensitivity of the closed loop system to external disturbances increases with N [3], [4]. Third, there is a lack of design methods for decentralized architectures. For N vehicles, in general, N distinct controllers need to be designed, for which few control design methods exist. This has led to the examination of only the symmetric control among bidirectional architectures [4], [5], [6]. Some symmetry aided simplifications are possible for analysis and design in this case.

In this paper, we present an eigenvalue optimization based method that is targeted at these problems in the control of vehicular platoons. The method builds upon our earlier work [7], [2] where we presented a partial differential equation (PDE) approximation of the vehicular platoon problem. Using the PDE model, we showed that a small perturbation (asymmetry) in the controller gains from their nominal (symmetric) value can improve the closed-loop damping. In particular, the least stable eigenvalue approaches 0 as $O(\frac{1}{N})$ with mistuning, whereas it approaches 0 as $O(\frac{1}{N^2})$ in the symmetric bidirectional case. The analysis was carried out using a perturbation method for vanishingly small amounts of mistuning.

The success of the "small mistuning" approach in achieving large stability margin improvement naturally raises the following question: how to maximize the stability margin by designing the mistuning gains when potentially large deviations from the nominal symmetric gains are allowed (subject to some realistic constraints on the gains)? This paper attempts to address this question. The proposed method draws on eigenvalue optimization literature from PDEs that has a long history going back to Joseph Keller's seminal paper on solution to the Lagrange problem [8].

In this paper, we obtain rigorous $O(\frac{1}{N})$ estimate on the least stable eigenvalue as $N \to \infty$. The approximations implicit in the perturbation calculation are not present here and the conclusion that mistuning leads to an order of magnitude improvement of stability margin is exact. We also show that the distributed control gains (mistuning profile) obtained using optimization is consistent with the results of the perturbation calculation (that appeared in [7], [2]) in the limit that the mistuning size $\epsilon \to 0$. These results are also verified using numerical calculations with the spatially discrete platoon model. The numerical computations also show that the method leads to a design that not only has an improved stability margin – as measured by the least stable eigenvalue – but also a lower sensitivity to disturbances (compared to the symmetric bidirectional case).

The rest of the paper is organized as follows: Section II describes the discrete and continuous models of the platoon problem. Section III presents a review of the stability analysis in [2], [7]. The eigenvalue optimization is discussed in Section IV with numerical results in Section V.

II. DISCRETE AND CONTINUOUS MODELS WITH BIDIRECTIONAL CONTROL

A. Closed-loop platoon dynamics (discrete model)

Consider a platoon of N identical vehicles moving in a straight line as shown schematically in Figure 1(a). Let $Z_i(t)$ and $V_i(t) := \dot{Z}_i(t)$ denote the position and the velocity, respectively, of the ith vehicle for $i = 1, 2, \ldots, N$. Each vehicle is modeled as a double integrator:

$$\ddot{Z}_i = U_i + W_i, \tag{1}$$

where U_i is the control (engine torque) applied on the ith vehicle, and W_i is the disturbance acting on the ith vehicle.

Following [1], we introduce a fictitious lead vehicle and a fictitious follow vehicle, indexed as 0 and $N+1$ respectively. Their behavior is specified by imposing a constant velocity trajectories as $Z_0(t) = V_d t$ and $Z_{N+1} = V_d t - (N+1)\Delta$.

For the decentralized bidirectional linear control architecture, the control U_i for the ith vehicle is given by

$$U_i = k_i^{(f)}(Z_{i-1} - Z_i - \Delta) - k_i^{(b)}(Z_i - Z_{i+1} - \Delta) - b_i(V_i - V_d). \tag{2}$$

where $k_i^{(\cdot)}, b_i$ are positive constants. The superscripts (f) and (b) correspond to *front* and *back*, respectively.

To facilitate analysis, we consider a coordinate change

$$y_i = 2\pi\left(\frac{Z_i(t) - V_d t + L}{L}\right), \qquad v_i = 2\pi\frac{V_i - V_d}{L}, \tag{3}$$

where $L = (N+1)\Delta$ denotes the *desired platoon length* (see Figure 1(b)). The scaling ensures that $y_0(t) \equiv 2\pi$, $y_i(t) \in [0, 2\pi]$, and $y_{N+1}(t) \equiv 0$.

B. Continuous model of closed loop platoon dynamics

The PDE is derived with respect to a scaled spatial coordinate $x \in [0, 2\pi]$. The starting point is a continuous approximation:

$$v(x, t) := v_i(t) \quad \text{at } x = y_i.$$

Similarly, $b(x), k^{(f)}(x), k^{(b)}(x)$ are used to denote continuous approximations of discrete gains $b_i, k_i^{(f)}, k_i^{(b)}$ respectively. The PDE model, derived in [2], [7], describes the evolution of spatio-temporal velocity perturbation $v(x, t)$ along the platoon:

$$\left(\frac{\partial^2}{\partial t^2} + b(x)\frac{\partial}{\partial t}\right)v = \frac{1}{\rho_0}(k^{(f)}(x) - k^{(b)}(x))\frac{\partial v}{\partial x}$$
$$+ \frac{1}{2\rho_0^2}(k^{(f)}(x) + k^{(b)}(x))\frac{\partial^2 v}{\partial x^2} \tag{4}$$

where $\rho_0 \doteq \frac{N}{2\pi}$. It is shown in [2] that for any fixed value of N, a suitable finite-difference approximation of the PDE yields the dynamics of the discrete model.

Because of the fictitious lead and follow vehicles, the appropriate boundary condition is of the Dirichlet type:

$$v(0, t) = v(2\pi, t) = 0. \quad \forall t \in [0, \infty). \tag{5}$$

III. STABILITY ANALYSIS AND CONTROL DESIGN

The PDE is useful because it succinctly describes the spatial aspects of dynamics for the closed-loop vehicular platoon. In [2], [7], the PDE is used for stability analysis and the mistuning-based control design. These results are briefly reviewed next:

A. Stability analysis

With a symmetric bidirectional control, all the control gains are constant: $k^{(f)}(x) = k^{(b)}(x) \equiv k_0$ and $b(x) \equiv b_0$. The PDE (4) simplifies to

$$\left(\frac{\partial^2}{\partial t^2} + b_0\frac{\partial}{\partial t} - a_0^2\frac{\partial^2}{\partial x^2}\right)v = 0, \tag{6}$$

a damped wave equation with *wave speed* $a_0 := \frac{\sqrt{k_0}}{\rho_0}$. The eigenvalues are easily computed by taking the Laplace transform and we have the following result:

Lemma 1 (Corollary 1 in [2]): Consider the eigenvalue problem for the PDE (6) with Dirichlet boundary condition (5). The least stable eigenvalue, denoted by s_1^+, satisfies

$$s_1^+ = -\frac{\pi^2 k_0}{b_0}\frac{1^2}{N^2} + O\left(\frac{1}{N^4}\right),$$

in the limit as $N \to \infty$. □

The Lemma shows that the stability margin of a large platoon deteriorates as $O(\frac{1}{N^2})$, when the number of vehicles is large. The conclusion is independent of *any* constant values of the controller gains k_0 and b_0.

B. Mistuning based control design

A mistuning based control design was proposed in [2], [7] to improve the stability margin. The idea was to increase the modulus of the least stable eigenvalue by introducing small perturbations in forward and backward control gains:

$$k^{(f)}(x) = k_0 + \epsilon k^{(f,pert)}(x),$$
$$k^{(b)}(x) = k_0 + \epsilon k^{(b,pert)}(x), \tag{7}$$

where $\epsilon > 0$ is a small parameter signifying the amount of mistuning and $k^{(f,pert)}(x)$, $k^{(b,pert)}(x) \in L^2([0, 2\pi])$ describe *perturbation* from the nominal value k_0. Define

$$k_s(x) := k^{(f,pert)}(x) + k^{(b,pert)}(x),$$
$$k_m(x) := k^{(f,pert)}(x) - k^{(b,pert)}(x).$$

The mistuned version of the PDE (4) is then given by

$$\frac{\partial^2 v}{\partial t^2} + b_0\frac{\partial v}{\partial t} = L(\epsilon)v, \tag{8}$$

where

$$L(\epsilon)v := a_0^2\frac{\partial^2 v}{\partial x^2} + \epsilon\left[\frac{k_m(x)}{\rho_0}\frac{\partial v}{\partial x} + \frac{k_s(x)}{2\rho_0^2}\frac{\partial^2 v}{\partial x^2}\right]. \tag{9}$$

The objective of mistuning based control design is to obtain the *mistuning profiles* $k_s(x)$ and $k_m(x)$ that minimize the least stable eigenvalue. For this problem, we have the following asymptotic result:

Corollary 1 (Corollary 2 in [7]): Consider the problem of minimizing the least-stable eigenvalue of the PDE (8)

with Dirichlet boundary condition (5) by choosing a function $k_m(x) \in L^2([0, 2\pi])$ such that $\int_0^{2\pi} |k_m(x)|^2 dx = 1$. In the limit as $\epsilon \to 0$, the optimal mistuning profile is given by $k_m^*(x) = -\frac{1}{\sqrt{\pi}} \sin(x)$. With this profile, the least stable eigenvalue is

$$s_1^+(\epsilon) = -\frac{\epsilon \sqrt{\pi}}{2b_0} \frac{1}{N} + O(\epsilon^2) + O(\frac{1}{N^2})$$

in the limit as $\epsilon \to 0$ and $N \to \infty$. $\qquad \square$

The corollary shows that the asymptotic properties of the closed-loop stability margin can be improved even with an *arbitrarily small perturbation*. With mistuning the least stable eigenvalue approaches zero only as $O(\frac{1}{N})$ as opposed to $O(\frac{1}{N^2})$ without mistuning.

The reader is referred to [7], [2] for the details.

IV. EIGENVALUE OPTIMIZATION BASED CONTROL DESIGN

The perturbation based analysis and the resulting control design procedure works well in practice (see numerical results both in this paper and in [7], [2]). However, one can provide guarantees only in the limit of small ϵ. Moreover, the spatial profiles predicted by these calculations are optimal only for vanishingly small values of ϵ. This motivates the topic of this paper - to develop an approach to optimize the least stable eigenvalue directly.

In this section we seek to determine the optimal mistuning profiles for $k_m(x)$ and $k_s(x)$ given some $\epsilon > 0$. On taking a Laplace transform of the PDE (8) with respect to the time variable, one easily obtains the characteristic equation for the least stable eigenvalue

$$s^2 + b_0 s = \lambda_1, \qquad (10)$$

where λ_1 is the principal (with largest real part) eigenvalue of the elliptic operator $L(\epsilon)$ (defined in (9)). By a standard argument in Strum-Louiville (S-L) theory, λ_1 is real with a positive eigenfunction [9]. As a result of (10), the problem of minimizing the least stable eigenvalue of the PDE (8) is equivalent to minimizing λ_1, the principal (with largest real part) eigenvalue of $L(\epsilon)$ by choosing the functions $k_m(s), k_s(x) \in L^2$. Using the characteristic equation, it also follows that

$$s_1^+ = \frac{\lambda_1}{b_0} \qquad (11)$$

in the limit as $N \to \infty$. In the following, we thus focus on minimizing λ_1 by choosing the functions $k_m(s), k_s(x) \in L^2([0, 2\pi])$. The least stable eigenvalue can then be obtained by solving (10) or simply using the asymptotic formula (11).

For problem of minimizing λ_1 to be well-posed, an additional constraint on $k_m(x)$ and $k_s(x)$ is needed. In the following, we impose the constraints

$$k_s(x) \equiv 0 \text{ and } \|k_m\|_{L^2} = 1. \qquad (12)$$

where $\|k_m\|_{L^2}^2 = \int_0^{2\pi} k_m(x)^2 dx$. The constraint $k_s(x) = 0$ is assumed for the sake of simplicity of the presentation and because it appears as part of the coefficient $\frac{1}{N^2} k_s(x)$

in (9). Any improvement due to $k_s(x)$ alone is $O(\frac{1}{N^2})$ while $k_m(x)$ can potentially deliver an $O(\frac{1}{N})$ shift in eigenvalue location. This is also reflected in estimates obtained using the perturbation methods (see Corollary 1).

Thus, the problem of minimizing the least stable eigenvalue of the PDE (8) is converted to the following optimization problem:

$$\min_{\{k_s(x) \equiv 0, \|k_m(x)\|_{L^2} = 1\}} \lambda_1. \qquad (13)$$

Even with $k_m(x)$ alone, the optimization of a non self-adjoint eigenvalue problem (as in our case) is challenging with limited theory for guidance; see Section 16 of the review paper [10] on the topic of eigenvalue optimization.

As a first step, we relax the optimization problem by replacing the operator $L(\epsilon)$ by its self-adjoint (symmetric) component:

$$L^s(\epsilon)\eta = (\frac{L + L^*}{2})\eta = a_0^2 \frac{d^2\eta}{dx^2} - \frac{\epsilon}{2\rho_0} k_m'(x)\eta, \qquad (14)$$

where L^* is the adjoint of L and $k_m'(x) := \frac{dk_m}{dx}(x)$. Let λ_1^s denote the principal eigenvalue of L^*:

$$a_0^2 \frac{d^2\phi}{dx^2} - \frac{\epsilon}{2\rho_0} k_m'(x)\phi = \lambda^s \phi \qquad (15)$$

with Dirichlet boundary conditions $\phi(0) = \phi(2\pi) = 0$. The following lemma gives the relationship between λ_1^s and λ_1.

Lemma 2: Let λ_1 denote the principal eigenvalue of the operator L in (9) and λ_1^s denote the principal eigenvalue of the self-adjoint operator L^s in (14). Then,

$$\lambda_1 \leq \lambda_1^s. \qquad \square$$

Proof of Lemma 2. Let λ_1 be the principal eigenvalue and $\phi(x)$ be the corresponding positive eigenfunction of the non self-adjoint problem:

$$a_0^2 \frac{d^2\phi}{dx^2} + \epsilon \frac{k_m(x)}{\rho_0} \frac{\partial \phi}{\partial x} = \lambda_1 \phi.$$

Multiplying by ϕ and integrating by parts, we obtain

$$-a_0^2 \int_0^{2\pi} \left(\frac{d\phi}{dx}\right)^2 dx - \frac{\epsilon}{2\rho_0} \int_0^{2\pi} k_m'(x)\phi^2 dx = \lambda_1 \int_0^{2\pi} \phi^2 dx.$$

We have

$$\lambda_1 \leq \max_{\phi > 0} \frac{\left[-a_0^2 \int \left(\frac{d\phi}{dx}\right)^2 dx - \frac{\epsilon}{2\rho_0} \int k_m'(x)\phi^2 dx\right]}{\int \phi^2 dx} = \lambda_1^s,$$

where the last equality follows from the variational characterization of the principal eigenvalue for a self-adjoint elliptic problem [9]. $\qquad \blacksquare$

Therefore, instead of the original eigenvalue optimization problem (13), we pose and solve the following simpler optimization problem:

$$\min_{\{k_m(x): \|k_m\|_{L^2} = 1\}} \lambda^s \qquad (16)$$

where λ^s is the principal (largest) eigenvalue of L^s. Note that we have dropped the subscript due to notational convenience. Because of Lemma 2, we can bound (from above) the solution of the original problem (13) in terms of solution of the self-adjoint problem (16).

We obtain the solution of the self-adjoint eigenvalue optimization problem (16) using a variational method originally due to Keller [8], which is presented in the following Lemma:

Lemma 3: Consider the eigenvalue optimization problem (16). The optimal mistuning profile is

$$k_m^*(x) = -C\phi(x)\phi'(x), \qquad (17)$$

where C is a constant and ϕ is the principal eigenfunction of the nonlinear BVP

$$a_0^2 \phi'' + \frac{\epsilon C}{2\rho_0}\left[\phi\phi'' + (\phi')^2\right]\phi = \lambda^s\phi. \qquad (18)$$

with $\phi(0) = \phi(2\pi) = 0$; $\phi'(x) \doteq \frac{d\phi}{dx}(x)$ and $\phi''(x) \doteq \frac{d^2\phi}{dx^2}(x)$. □

Proof. Assume $k_m^*(x)$ minimizes the largest eigenvalue and is thus the solution. After Keller [8], we introduce a family of functions $k_m(x, \delta)$ with $k_m^*(x) = k_m(x, 0)$ to construct a differential characterization of this optimal. For each δ, the principal eigenvalue and the eigenfunction are given by $\lambda^s(\delta)$ and $\phi(x, \delta)$ respectively. Differentiating (15) with respect to δ and evaluating at $\delta = 0$ gives

$$a_0^2 \frac{d^2\phi_\delta}{dx^2} - \frac{\epsilon}{2\rho_0}k_m'(x)\phi_\delta - \frac{\epsilon}{2\rho_0}(k_m')_\delta(x)\phi = \lambda^s\phi_\delta,$$

where $(k_m')_\delta(x) = \frac{\partial(k_m')(x)}{\partial\delta}|_{\delta=0}$ and $\phi_\delta = \frac{\partial\phi}{\partial\delta}|_{\delta=0}$. Multiplying by ϕ, integrating, and using (15) gives

$$\int_0^{2\pi}(k_m')_\delta(x)\phi^2 dx = 0 \qquad (19)$$

On differentiating the constraint (12), we obtain

$$\int_0^{2\pi}k_m(x)(k_m)_\delta(x)dx = 0. \qquad (20)$$

Since $(k_m)_\delta(x)$ represents an arbitrary perturbation about the optimal, the two equations (19)-(20) imply that the optimal mistuning profile is given by

$$k_m^*(x) = -C\phi(x)\frac{d\phi}{dx}(x), \qquad (21)$$

where C is some constant. It follows that for this optimal mistuning profile

$$k_m'(x) = -\frac{C}{2}\frac{d^2(\phi)^2}{dx^2}.$$

Substituting this in (15), one obtains the nonlinear BVP (18). ∎

As a result of the Lemma, control gains can be obtained by solving the nonlinear BVP (18).

We consider the $\epsilon \to 0$ limit first. In this limit, the principal eigenfunction (of (18)) is given by $\phi = \sin(\frac{x}{2})$. Using (17),
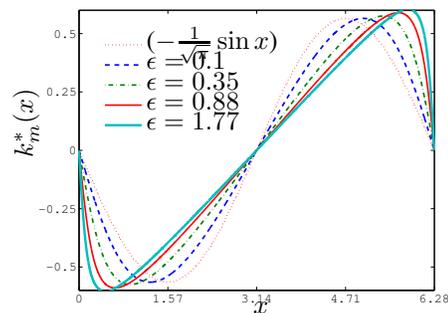

Fig. 2. The optimal mistuning pattern $k_m^*(x)$ computed according to the procedure laid out in section VI-A for three different values of ϵ. The parameters are $N = 50$ and $k_0 = 1$.

one immediately obtains the optimal mistuning pattern for the limiting case $\epsilon \to 0$:

$$k_m^*(x) = -C\sin(x),$$

where $C = \frac{1}{\sqrt{\pi}}$ satisfies the norm constraint. This is consistent with the optimal mistuning profile obtained using the perturbation method (see Corollary 1). For small ϵ, this also provides an estimate of the eigenvalue

$$\lambda^s = -\frac{a_0^2}{4} - \frac{\epsilon}{16\pi\rho_0} + O(\epsilon^2),$$

which using (11) gives the estimate of Corollary 1.

For the general case, we solve the BVP (18). The details of the calculation appear in the Appendix section VI-A where we show that

$$\sqrt{-\lambda^s} = \frac{\sqrt{k_0}}{2N}\int_0^{2\pi}\left[1 + \frac{\epsilon\beta N}{4\pi k_0}\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}}d\theta, \qquad (22)$$

where β is a positive constant that is independent of N, and recall k_0 is the nominal symmetric value of control gain.

Using the asymptotic formula in conjunction with the bound in Lemma 2, one has the following corollary that provides a rigorous $O(\frac{1}{N})$ bound on the least stable eigenvalue of the PDE (8).

Corollary 2: Let s_1^+ denote the least stable eigenvalue of the PDE (8) with Dirichlet boundary condition (5). With the choice of mistuning profile given in Lemma 3, the least stable eigenvalue is given by the asymptotic formula

$$s_1^+ = -\frac{\epsilon\beta}{\pi b_0 N} + O(\frac{1}{N^2}), \qquad (23)$$

as $N \to \infty$ for any fixed $\epsilon > 0$; β is a constant that is independent of N. ∎

V. NUMERICAL RESULTS

Figure 2 depicts the optimum mistuning profiles for the self-adjoint PDE for three different values of ϵ. Consistent with the results obtained using the perturbation method, the optimal mistuning profile is close to the sinusoidal pattern for small values of ϵ.
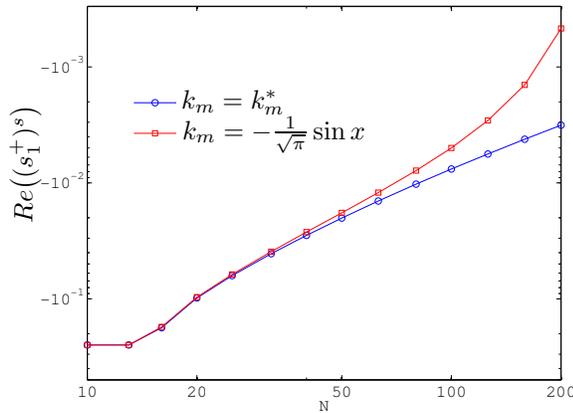
Fig. 3. Real part of the least stable eigenvalue of the self-adjoint PDE (24) with the optimal mistuning profile and sinusoidal profile. The parameters are $k_0 = 1$, $b_0 = 0.5$, and $\epsilon \approx 0.35$. As predicted, the optimal mistuning profile results in a smaller eigenvalue compared to that from a sinusoidal one.
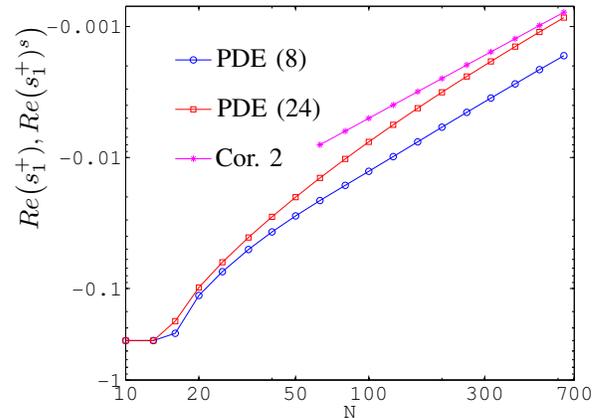

Fig. 4. The least stable eigenvalue of the governing PDE (8) and of the self-adjoint PDE (24) with $k_m(x) = k_m^*(x)$, and the predictions of Corollary 2 for the self-adjoint PDE for $N > 50$. The parameters are $k_0 = 1$, $b_0 = 0.5$, and $\epsilon \approx 0.35$. Consistent with Lemma 2, the least stable eigenvalue of the self-adjoint PDE upper bounds that of the original non self-adjoint problem. Moreover, Corollary 2 predicts the least stable eigenvalue of the self-adjoint PDE quite accurately for large N.

Figure 3 depicts the trend of the least stable eigenvalue of the self-adjoint PDE:

$$\frac{\partial^2 v}{\partial t^2} + b_0 \frac{\partial v}{\partial t} = L^s(\epsilon)v, \qquad (24)$$

for both sinusoidal and $k_m^*(x)$ mistuning. The least stable eigenvalue of (24) is denoted by $(s_1^+)^s$. It is seen from the figure that the optimal mistuning profile $k_m^*(x)$ indeed results in a smaller least stable eigenvalue than the sinusoidal one.

Figure 4 depicts the least stable eigenvalue of the self-adjoint PDE with the optimal mistuning profile k_m^* and that of the original non self-adjoint PDE (8) with the same mistuning applied to it. The figure also plots the predictions of Corollary 2 for large values of N, in particular, for $N > 50$. As seen from the plot, the predicted eigenvalues of the self-adjoint PDE match quite accurately the numerically computed ones. In addition, as predicted by Lemma 2, the least stable eigenvalue of the self-adjoint PDE upper bounds that of the original non self-adjoint problem. This plot provides numerical evidence of the rigorous $O(\frac{1}{N})$ bound for the eigenvalues of the original PDE (8).

Numerically, the optimal mistuning profile obtained for self-adjoint PDE was found to be sub-optimal for the non self-adjoint PDE corresponding to the discrete platoon. In particular, for the values of ϵ tested and shown in the figures, the sinusoidal mistuning profile was seen to provide greater damping for the platoon. This is seen from the plots in Figure 5. The plot nevertheless shows distinct improvement in stability margin resulting from mistuned control compared to the symmetric bidirectional control.

In addition, Figure 5 presents numerical corroboration that the predictions of the PDE model match the predictions of the state-space model of the platoon dynamics accurately. The least stable eigenvalue of the platoon model (the state matrix for (1)-(2)) was computed by choosing the controller gains according to the mistuning profile, with $k_0 = 1$, $b_0 = $

0.5, and $\epsilon \approx 0.35$. The gains for the sinusoidal mistuning profile are obtained from sampling $k^{(f)}(x) = 1 - 0.1 \sin(x)$ and $k^{(b)}(x) = 1 + 0.1 \sin(x)$. For the optimal profile, the gains are chosen by sampling $k_f(x) = 1 + \epsilon k_m^*(x)/2$ and $k^{(f)}(x) = 1 - \epsilon k_m^*(x)/2$. In both cases, the gains differed from the nominal gain $k_0 = 1$ by $\pm 10\%$. The figure shows that the eigenvalues computed from the PDE (8) predict the eigenvalues of the discrete platoon model accurately over a range of values of N, for both types of mistuning profiles.

A. Sensitivity to Disturbance

In this section, we present results on the closed-loop's sensitivity to disturbances with mistuned controllers. Automated platoons suffer from high sensitivity to external disturbances; this is referred to as "string instability" or "slinky-type effects" [11], [3]. The H_∞ norm of the transfer function G_{we} from the disturbance $\mathbf{w} = [W_1, \ldots, W_N]$ to the inter-vehicle spacing error $\mathbf{e} = [e_1, \ldots, e_N]$, with $e_i \doteq (Z_{i-1} - Z_i - \Delta)$ is a measure of the closed loop's sensitivity to external disturbances.

Figure 6 summarizes the H_∞ norm of the transfer function G_{we}. The figure shows that mistuning helps in reducing the sensitivity to disturbances. Both the sinusoidal mistuning profile and the optimal profile (for the self-adjoint PDE) has this beneficial effect. The optimal mistuning profile designed for the self-adjoint PDE was seen to reduce the sensitivity to disturbances more than the sinusoidal mistuning does. Further analysis of the effect of mistuning on sensitivity to disturbances is a subject of future work.

VI. APPENDIX

A. Solution of the nonlinear BVP

We begin by observing that the nonlinear BVP (18) admits a symmetry whereby if $\phi(x)$ is an eigenfunction then so is
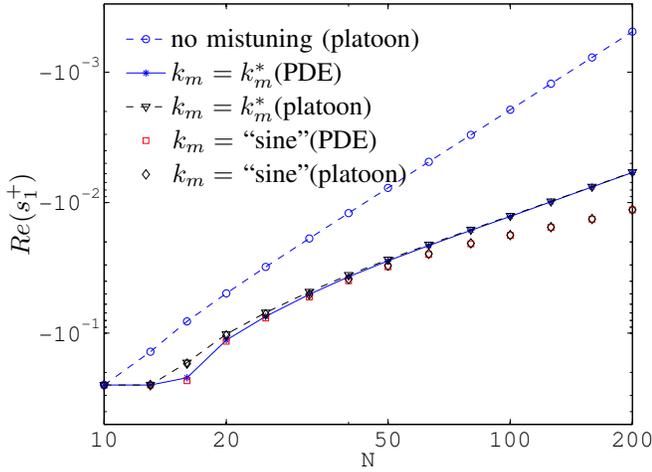
Fig. 5. Least stable eigenvalues of the mistuned PDE (8) as well as that of the discrete platoon with sinusoidal mistuning profile, and k_m^* mistuning profile (optimal for the self-adjoint problem (24)). The legend "(platoon)" means that the eigenvalues are those of the state matrix for (1)-(2), whereas "(PDE)" means that the eigenvalues are of the governing PDE (4). Also shown are the least stable eigenvalues of the closed loop platoon without any mistuning (i.e., symmetric bidirectional control). "sine" denotes $k_m(x) = -\frac{1}{\sqrt{\pi}} \sin(x)$. The parameters are $k_0 = 1$, $b_0 = 0.5$, and $\epsilon = 0.35$, which results in the mistuned gains differing from the nominal (symmetric bidirectional) case by $\pm 10\%$. The eigenvalues of the discrete platoon match the eigenvalues of the discrete platoon accurately, but the optimal profile designed for the self-adjoint problem does not perform as well for the original non self-adjoint problem as the sinusoidal profile.

$\phi(2\pi - x)$. This implies that $\phi(x) = \phi(2\pi - x)$, and at $x = \pi$

$$\frac{d\phi}{dx}(\pi) = 0. \tag{25}$$

To obtain the solution, the ODE (18) is first simplified to

$$\frac{\phi'}{\lambda^s - \frac{\epsilon C}{2\rho_0}(\phi')^2} d\phi' = \frac{\phi}{a_0^2 + \frac{\epsilon C}{2\rho_0}\phi^2} d\phi,$$

that on integration gives

$$\lambda^s - \frac{\epsilon C}{2\rho_0}(\phi')^2 = \frac{D}{a_0^2 + \frac{\epsilon C}{2\rho_0}\phi^2}, \tag{26}$$

where D is a constant of integration. Using (25), we get $D = \lambda^s \left(a_0^2 + \frac{\epsilon C}{2\rho_0}y_0^2\right)$ where $y_0 := \phi(\pi)$. As a result, (26) becomes

$$\left[1 + \frac{\epsilon C}{2\rho_0 a_0^2}\phi^2(x)\right]\left[1 - \frac{\epsilon C}{2\rho_0 \lambda^s}(\frac{d\phi}{dx})^2\right] = \left[1 + \frac{\epsilon C}{2\rho_0 a_0^2}y_0^2\right],$$

where $y_0 = \phi(\pi)$. After some manipulation, this equation leads to the integral

$$\frac{a_0}{\sqrt{-\lambda^s}} \int_0^{\phi(x)} \left[\frac{1 + \frac{\epsilon C}{2\rho_0 a_0^2}y^2}{y_0^2 - y^2}\right]^{\frac{1}{2}} dy = x. \tag{27}$$

The solution to this integral requires elliptic functions of the second kind. In particular, we propose a coordinate change

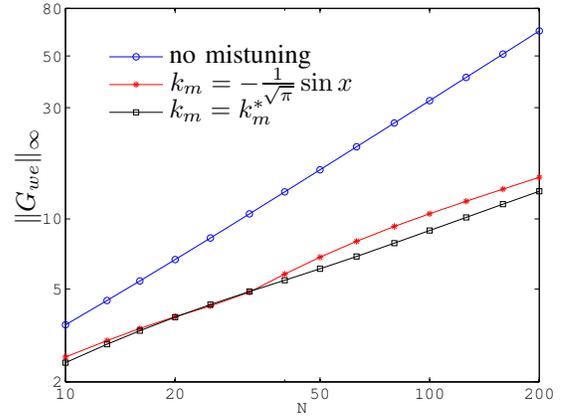$$\phi = y_0 \sin(\frac{\theta}{2}), \quad \theta \in [0, 2\pi]. \tag{28}$$

Fig. 6. H_∞ norm of the transfer function G_{we} from disturbance $\mathbf{w}$ to spacing error $\mathbf{e}$ for three cases: without mistuning, with sinusoidal mistuning, and with k_m^* mistuning. The H_∞ norms are significantly lower with mistuning than with none (i.e., symmetric bidirectional control), even though the mistuning was only $\pm 10\%$ ($\epsilon = 0.35$).

and define $\beta := C y_0^2$. Using (27), θ is a solution to an implicit elliptic integral equation

$$\int_0^\theta \left[1 + \frac{\epsilon \beta}{2\rho_0 a_0^2} \sin^2 \frac{\theta}{2}\right]^{\frac{1}{2}} d\theta = \frac{2\sqrt{-\lambda^s}}{a_0} x, \tag{29}$$

where substituting $\theta = \pi$ and using (28), one obtains an implicit relationship between β and λ^s:

$$\sqrt{-\lambda^s} = \frac{\sqrt{k_0}}{2N} \int_0^{2\pi} \left[1 + \frac{\epsilon \beta N}{4\pi k_0} \sin^2 \frac{\theta}{2}\right]^{\frac{1}{2}} d\theta, \tag{30}$$

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