Estimation From Relative Measurements: Electrical Analogy and Large Graphs

Prabir Barooah, *Member, IEEE*, and João P. Hespanha, *Fellow, IEEE*

***Abstract*—We examine the problem of estimating vector-valued variables from noisy measurements of the difference between certain pairs of them. This problem, which is naturally posed in terms of a measurement graph, arises in applications such as sensor network localization, time synchronization, and motion consensus. We obtain a characterization on the minimum possible covariance of the estimation error when an arbitrarily large number of measurements are available. This covariance is shown to be equal to a matrix-valued effective resistance in an infinite electrical network. Covariance in large finite graphs converges to this effective resistance as the size of the graphs increases. This convergence result provides the formal justification for regarding large finite graphs as infinite graphs, which can be exploited to determine scaling laws for the estimation error in large finite graphs. Furthermore, these results indicate that in large networks, estimation algorithms that use small subsets of all the available measurements can still obtain accurate estimates.**

***Index Terms*—Distributed estimation, electrical networks, infinite dimensional systems, parameter estimation, sensor networks.**

I. Introduction

We consider the estimation of vector-valued variables based on noisy measurements of the difference between certain pairs of such variables. In particular, denoting the variables of interest by $\{x_u : u \in \mathbf{V}\}$ where $\mathbf{V} := \{1, 2, \ldots\}$, we consider problems for which noisy "relative measurements" of the form

$$\zeta_{u,v} = x_u - x_v + \epsilon_{u,v} \tag{1}$$

are available, where $\epsilon_{u,v}$ denotes measurement noise. The ordered pairs of indices (u, v) for which we have relative measurements form a set $\mathbf{E}$ that is a (typically strict) subset of the set $\mathbf{V} \times \mathbf{V}$ of all pairs of indices. Just with relative measurements, the x_us can be determined only up to an additive constant. To avoid this ambiguity, we assume that a particular variable (say x_o) is used as the *reference* with x_o known. The problem of interest is to estimate the node variables from all the available measurements. This estimation problem is relevant to several sensor and multi-agent network applications, such as localization with noisy distance and angle measurement [1]–[3], time-synchronization [2], [4], [5], and motion-coordination [6]; see [6] and [7] for an overview. The estimation problem we study is an instance of a general class of parameter estimation problems in sensor networks called *self-calibration* [3], [8]. Among the applications aforementioned, localization in sensor networks has probably attracted the most attention in recent times. The reader is advised, however, that localization from range-only measurements, such as those considered in [9] and [10], do not fall into the problem category investigated in this paper. The measurement equations(1) can be expressed in terms of a directed graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with an edge from node u to v if the measurement $\zeta_{u,v}$ is available. The graph $\mathcal{G}$ is called the *measurement graph*, and each vector x_u, $u \in \mathbf{V}$ is called the uth *node variable*.

The main result of this paper relates to infinite sets of available measurements, which is used to model the limiting case for a very large number of measurements. When the number of measurements is infinite, we show that for every positive constant $\epsilon > 0$, it is possible to construct an unbiased estimate for a node variable x_u that uses only a finite subset of the available measurements but whose estimation error variance is only ϵ above the minimum possible estimation error variance that could be obtained by considering the whole infinite set of measurements. The main assumption needed is that the graph must have a finite maximum node degree, i.e., that there is a maximum number of relative measurements involving each node variable. An implication of this result is that for estimation problems based on relative measurements, after a certain point, considering more measurements will only marginally improve the quality of the estimate. On the positive side, this simplifies the construction of estimation algorithms in large-scale networks because it justifies considering a relatively small subset of measurements. Distributed algorithms to estimate the node variables from relative measurements have been examined in [2] and [7] in which nodes with embedded processing and communication capability estimates their variables by local computation and communication. Although the algorithms in [2] and [7] were developed for finite graphs, in a large graph these algorithms may take a long time to provide accurate estimates, since the information about all the available measurements are fused iteratively to determine the estimates. The results of this paper suggest that it may be possible to devise algorithms such that they obtain estimates quite fast, while sacrificing little accuracy.

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P. Barooah was with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA. He is now with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611 USA (e-mail: pbarooah@ufl.edu).

J. P. Hespanha is with the Center for Control, Dynamical Systems, and Computation, and the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: hespanha@ece.ucsb.edu).

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Another contribution of this paper is that the results established here help determine how the estimation error grows as the network size increases. It is often easier to establish asymptotic results on the minimum error variance for infinite graphs than for large finite graphs, since boundary effects are usually weaker in infinite graphs than in finite graphs. Scaling laws for the minimum estimation error in infinite graphs are investigated in [11]. The main convergence result of this paper provides the formal justification for regarding infinite graphs as suitable proxies for very large but finite graphs. It was assumed in [11] that estimation error covariances in infinite graphs are well defined. We show in this paper that this assumption is indeed valid, if certain properties are satisfied (those stated in Theorem 1). Therefore, the results in this paper also establish the conditions under which approximation of large finite graphs by infinite graphs is valid.

The key technical tool used to prove the results discussed above is the analogy between the measurement network and an electrical network. When the node variables are scalars and the measurement graph is finite, the variance of the estimation error for a node variable x_u is numerically equal to the effective resistance between the node u and the reference node o of an electrical network obtained from the measurement graph by placing at each edge $(u,v) \in \mathbf{E}$ a resistor whose resistance is equal to the variance of the measurement error $\epsilon_{u,v}$. This *electrical analogy* was noted by [4] for the time-synchronization problem.

In this paper, we show that an electrical analogy still holds for vector-valued node variables, provided that we consider *generalized electrical networks* in which currents, voltages, and resistors are matrix-valued, but still satisfy appropriately adapted forms of Kirchhoff's and Ohm's laws. In this case, the electrical network is obtained by placing at each edge $(u,v) \in \mathbf{E}$ a resistor whose (matrix-valued) resistance is equal to the covariance matrix of the measurement noise $\epsilon_{u,v}$. We further show that, when the measurement graph is infinite, as one considers increasingly large but *finite* subsets of the measurements, the covariance matrix of the estimation error of a node variable x_u converges to the (matrix-valued) effective resistance between node u and the reference node o of an infinite generalized electrical network obtained from the *infinite* measurement graph.

For certain infinite measurement graphs, such as d-D square lattices, the effective resistances can be explicitly computed. Because of the electrical analogy and the convergence result established in this paper, one can determine the smallest estimation error variance that could be obtained by considering the whole (infinite) set of measurements in such graphs. In practice, measurement graphs may not be lattices, but it is generally possible to embed them in lattices or find lattices that can be embedded in measurement graphs [11], [12]. It turns out that estimation error variances are monotonic with respect to the partial order defined by graph embedding, which is a consequence of our extension of the so-called Rayleigh's Monotonicity Law [12] to generalized electrical networks. As a consequence, we can construct upper and lower bounds on the estimation error variances in these measurement graphs from the results available for lattices. A preliminary study on establishing scaling laws for the estimation error variance in large graphs using embedding in lattices was undertaken in [11].

Numerical studies on subgraphs of the 2-D lattice show that information from a relatively small finite subgraph of the infinite measurement graph is sufficient to provide an estimate whose variance is quite close to the minimum variance that is achievable by using all the measurements. Increasing the size of the subgraph, i.e., increasing the number of measurements processed beyond a certain point does not yield a commensurate return in the decrease in variance. In particular, when constructing the estimate of a particular node us variable, if all measurements involving nodes lying within a distance of twice the distance between u and the reference are used, the difference between the resulting estimation error variance of x_u and the minimum possible variance is less than 10%. For an arbitrary infinite graph, similar trends are expected as long as the graph is close to a lattice in an appropriate sense. The question of what is a meaningful measure for a graph to be close to a lattice for the estimation problem has been addressed in [11].

The rest of the paper is organized as follows. Section II summarizes the main result of the paper. Section III introduces generalized electrical networks. Section IV establishes the analogy between the estimation and electrical network problems for finite measurement graphs and uses this to prove the main result. Section V describes scaling laws for the minimum possible estimation error in lattices and a numerical study on the convergence of the estimation error covariances in finite subgraphs as the subgraphs are increased in size. The paper concludes with a summary in Section VI.

II. Main Result

Consider a set of vector-valued variables $x_u \in \mathbb{R}^k, u \in \mathbf{V} := \{1,2,\ldots\}$, where the set $\mathbf{V}$ is either finite, or infinite but countable. These variables are to be estimated based on noisy relative measurements of the form

$$\zeta_{u,v} = x_u - x_v + \epsilon_{u,v}, \quad (u,v) \in \mathbf{E} \tag{2}$$

where $\epsilon_{u,v}$ denotes a zero-mean measurement noise and E is the set of pairs (u,v) for which relative measurements are available. The covariance matrix of the error $\epsilon_{u,v}$ is denoted by $P_{u,v} := \mathrm{E}\left[\epsilon_{u,v}\epsilon_{u,v}^T\right]$, where $\mathrm{E}(\cdot)$ denotes expectation and $(\cdot)^T$ denotes transpose. The measurement error covariance matrices are assumed to be positive definite and finite. The measurement errors on different edges are uncorrelated, i.e., for two edges $e, \bar{e} \in \mathsf{E}$, $\mathrm{E}[\epsilon_e \epsilon_{\bar{e}}^T] = 0$ unless $e = \bar{e}$. We assume that the value of a particular *reference variable* x_o is known and without loss of generality we take $x_o = 0$.

The accuracy of a node variable's estimate, measured in terms of the covariance of the estimation error, depends on the graph $\mathcal{G}$ as well as the measurement error covariances. The estimation problem is therefore formulated in terms of a *network* $(\mathcal{G}, P)$ where $P : \mathbf{E} \to \mathbb{S}^{k+}$ is a function that assigns to each edge $(u,v) \in \mathbf{E}$ the error covariance matrix $P_{u,v}$ of the measurement error associated with the edge (u,v). The symbol $\mathbb{S}^{k+}$ denotes the set of $k \times k$ symmetric positive definite matrices.

We are interested in problems for which the set of variables and available measurements is very large. We model this situation by making the number of variables and measurements countably infinite. The measurement graph $\mathcal{G}$ is therefore an

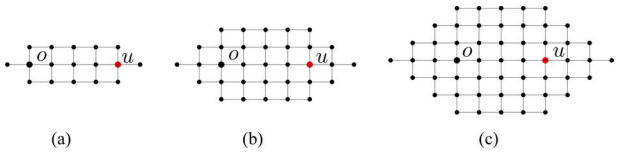


Fig. 1. A nested sequence of measurement graphs that "tend to" the 2-D square lattice.

infinite graph. In such problems, the question arises if it is possible to construct an estimate of an arbitrary node variable x_u by using only a finite subset of measurements such that its error covariance is arbitrarily close to the minimum error covariance achievable by using all the infinite number of measurements. To pose this question, we focus on an arbitrary node $u \in \mathbf{V}$—hereafter called the *node of interest*—and examine the estimates of x_u using larger and larger "subgraphs" as described next. First we recall certain graph theoretic terminology. A *undirected path* between a pair of nodes u_1 and u_N in a graph is a finite, alternating sequence of nodes and edges $u_1, e_1, u_2, e_2, \ldots, e_{N-1}, u_N$ such that every edge is incident on its two adjacent nodes in the sequence, and no node or edge is repeated. An edge (u, v) is said to be *incident* on the nodes u and v. A directed graph is said to be *weakly connected* if there is an undirected path between every pair of nodes. For two graphs $\mathcal{G}_1 = (\mathbf{V}_1, \mathbf{E}_1)$, $\mathcal{G}_2 = (\mathbf{V}_2, \mathbf{E}_2)$, the notation $\mathcal{G}_1 \subset \mathcal{G}_2$ means $\mathbf{V}_1 \subset \mathbf{V}_2$ and $\mathbf{E}_1 \subset \mathbf{E}_2$. We now consider a sequence of *finite* measurement subgraphs $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \mathcal{G}^{(3)}, \ldots$ that satisfies the following assumption.

Assumption 1 (Nested Sequence): The sequence of finite graphs $\mathcal{G}^{(1)}, \mathcal{G}^{(2)}, \mathcal{G}^{(3)}, \ldots$ has the following properties:

1. The sequence is *nested* in the sense that
$$\mathcal{G}^{(1)} \subset \mathcal{G}^{(2)} \subset \mathcal{G}^{(3)} \subset \cdots \subset \mathcal{G}.$$
2. The sequence *converges* to the graph $\mathcal{G}$ in the sense that every node and edge in $\mathcal{G}$ appears in one of the $\mathcal{G}^{(n)}$ for some finite n.
3. Each finite graph $\mathcal{G}^{(n)}$, $n \in \mathbb{N}$ is weakly connected.

□

In constructing such a nested sequence of finite graphs, every graph $\mathcal{G}^{(n)}$ should contain the reference node o and the node of interest u. Fig. 1 shows the first few elements of such a nested graph sequence that will eventually converge to the 2-D square lattice (the formal definition of a lattice will be provided in Section V-A). One could regard each finite subgraph $\mathcal{G}^{(n)}$ as describing a finite subset of available measurements that could be processed up to some time $t_n < \infty$ to construct an estimate of x_u. As time increases, more measurements can be processed, and, therefore, at some time $t_{n+1} > t_n$, the subgraph $\mathcal{G}^{(n+1)}$ contains more measurements than $\mathcal{G}^{(n)}$. In this context, we are interested in studying if there is a point after which there is little gain in processing more measurements, as this will not improve the estimate of x_u significantly. Essentially, we are asking whether or not the sequence of estimates produced using the nested sequence of subgraphs converges.

Given a finite subset of measurements characterized by one of the graphs $\mathcal{G}^{(n)}$, it is straightforward to compute the best linear unbiased (BLU) estimate $\hat{x}_u^{(n)}$ that minimizes the estimation error variance among all linear unbiased estimators. This estimate is a linear combination of the measurements $\zeta_e, e \in \mathbf{E}$ specified by a set of appropriately chosen coefficient matrices. In particular, the BLU estimate is given by

$$\hat{x}_u^{(n)} = \sum_{e \in \mathbf{E}^{(n)}} C_e^{(n)T} \zeta_e \tag{3}$$

where the function $C^{(n)} : \mathbf{E}^{(n)} \to \mathbb{R}^{k \times k}$ specifies the coefficients of the measurements. Note that in the equation above, and in the sequel, for a function f with the edge set E as the domain, we use f_e to denote the value of the function at an edge $e \in \mathbf{E}$. We call the function $C^{(n)}$ the *BLU estimator for x_u based on the finite graph $\mathcal{G}^{(n)}$*. Every estimator $C^{(n)}$ can be viewed as an element of the real linear vector space $\mathcal{H}_P$ consisting of all *edge-functions* of the form $C : \mathbf{E} \to \mathbb{R}^{k \times k}$ for which

$$\|C\|^2 := \sum_{e \in \mathsf{E}} \operatorname{Tr}\left(C_e^T P_e C_e\right) < \infty \tag{4}$$

where $\operatorname{Tr}(\cdot)$ denotes trace, and each P_e denotes the error covariance matrix for the measurement associated with the edge $e \in \mathsf{E}$. It is straightforward to show that $\mathcal{H}_P$ is a Hilbert space with the associated inner product $\langle C, \bar{C} \rangle = \sum_{e \in \mathbf{E}} \operatorname{Tr}\left(C_e^T P_e \bar{C}_e\right)$, $\forall\, C, \bar{C} \in \mathcal{H}_P$. We say that an edge-function in $\mathcal{H}_P$ has finite support if it has only a finite number of nonzero entries. Since all the sets $\mathbf{E}^{(n)}$ in (3) are finite, every estimator $C^{(n)}$ is a finite-support edge-function in $\mathcal{H}_P$.

For infinite graphs, the summation in (4) is actually a series. However, the series is absolutely convergent due to the positive definiteness of the P_e's, hence the order of the summation is immaterial and, therefore, the expression in (4) is well defined.

We now state the main result of the paper, which establishes the convergence of BLU estimators as $n \to \infty$. To state this result we recall that the *degree of a node* is defined as the number of edges that are incident on the node.

Theorem 1 (BLU Convergence): Consider a network $(\mathcal{G}, P)$ for which the measurement graph $\mathcal{G}$ has a finite maximum node degree, and for which the error covariance function P is uniformly bounded in the sense that there exist constant matrices $P_{\min}, P_{\max} \in \mathbb{S}^{k+}$ such that

$$P_{\min} \leq P_e \leq P_{\max}, \quad \forall\, e \in \mathbf{E}$$

where $A \leq B$ means that $A - B$ is negative semidefinite. For every node $u \in \mathbf{V} \backslash \{o\}$, where o is the reference node, if $\{\mathcal{G}^{(n)}\}$ is a nested sequence of finite graphs that satisfies Assumption 1 with u and o belonging to every graph in the sequence, the following statements hold.

1) The sequence of BLU estimates $\left\{\hat{x}_u^{(n)}\right\}$ converges in the mean-square sense.
2) The sequence of BLU estimators $\{C^{(n)}\}$ for x_u converges to some $C \in \mathcal{H}_P$.
3) The sequence of BLU estimation error co-variance matrices
$$\Sigma_{u,o}^{(n)} := \mathrm{E}\left[\left(x_u - \hat{x}_u^{(n)}\right)\left(x_u - \hat{x}_u^{(n)}\right)^T\right]$$

converges to a symmetric positive definite matrix $\Sigma_{u,o}$. Moreover, these BLU covariances converge monotonically in the sense that

$$\Sigma_{u,o}^{(1)} \geq \Sigma_{u,o}^{(2)} \geq \cdots \geq \Sigma_{u,o} > 0.$$

□

Theorem 1 shows that under the bounded degree assumption, by using only a finite number of measurements among the infinitely many potentially available, we can construct estimates whose error variance is arbitrarily close to the minimum possible variance that could be achieved by using all the available measurements. In addition, the estimates themselves converge and the "limiting" estimator is square-summable in the sense of(4).

Although Theorem 1 states that the BLU covariances in the finite graphs converges to a limiting covariance $\Sigma_{u,o}$, it does not specify what the limit is. However, the proof of this result actually provides a construction to obtain $\Sigma_{u,o}$ from the network $(\mathcal{G}, P)$ by showing that it is numerically equal to a matrix-valued *generalized effective resistance* in a generalized electrical network. This analogy with an electrical network is the key technical tool to prove Theorem 1 . It also provides additional intuition into the problem. Section III describes generalized electrical networks and presents the associated technical results needed to prove Theorem 1 .

III. Generalized Electrical Networks

A *generalized electrical network* $(\mathcal{G}, R)$ consists of a graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ (finite or infinite) together with a function $R : \mathbf{E} \to \mathbb{S}^{k+}$ that assigns to each edge $e \in \mathbf{E}$ a symmetric positive definite matrix R_e called the *generalized resistance* of the edge. A *generalized flow from node* $u \in \mathbf{V}$ *to node* $v \in \mathbf{V}$ *with intensity* $\mathbf{j} \in \mathbb{R}^{k\times k}$ is an edge-function $j : \mathbf{E} \to \mathbb{R}^{k\times k}$ such that

$$\sum_{\substack{(p,q)\in\mathbf{E}\\ p=\bar{p}}} j_{p,q} - \sum_{\substack{(q,p)\in\mathbf{E}\\ p=\bar{p}}} j_{q,p} = \begin{cases} \mathbf{j} & \bar{p} = u \\ -\mathbf{j} & \bar{p} = v \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad \forall\, \bar{p} \in \mathbf{V}. \tag{5}$$

We say that a flow i is a *generalized current* when there is a *node-function* $V : \mathbf{V} \to \mathbb{R}^{k\times k}$ for which

$$R_{u,v} i_{u,v} = V_u - V_v, \quad \forall\, (u,v) \in \mathbf{E}. \tag{6}$$

The node-function V is called a *generalized potential associated with the current* i. Equation (5) should be viewed as a generalized version of Kirchhoff's current law and can be interpreted as: the net flow out of each node other than u and v is equal to zero, whereas the net flow out of u is equal to the net flow into v and both are equal to the flow intensity $\mathbf{j}$. Equation (6) provides in a combined manner, a generalized version of Kirchhoff's loop law, which states that the net potential drop along a circuit must be zero, and Ohm's law, which states that the potential drop across an edge must be equal to the product of its resistance and the current flowing through it. A circuit is an undirected path that start and end at the same node. For $k = 1$, generalized electrical networks are the usual electrical networks with scalar currents, potentials, and resistors.

The following property for the generalized electrical networks is implicitly assumed throughout this section.

Assumption 2 (Generalized Electrical Network): The generalized electrical network $(\mathcal{G}, R)$ is constructed from a graph $\mathcal{G}$ that is weakly connected with a finite maximum node degree, and from an edge-resistance function R that is uniformly bounded, i.e., there exist constant symmetric positive-definite matrices $R_{\min}, R_{\max}$ such that $R_{\min} \leq R_e \leq R_{\max}, \forall\, e \in \mathbf{E}$.

The *energy dissipated* by an edge-function j in the network $(\mathcal{G}, R)$ is defined by

$$\|j\| := \left(\sum_{e\in\mathbf{E}} \operatorname{Tr}\left(j_e^T R_e j_e\right) \right)^{(1/2)}. \tag{7}$$

It is straightforward to verify that the set of edge-functions with finite dissipated energy constitutes a Hilbert space $\mathcal{H}_R$ with inner product $\langle j, \bar{j} \rangle = \sum_{e\in\mathbf{E}} \operatorname{Tr}\left(j_e^T R_e \bar{j}_e\right), \forall\, j, \bar{j} \in \mathcal{H}_R$. For infinite networks, the summation in (7) is an absolutely convergent series and the order of summation is irrelevant. Flows of finite support always belong to $\mathcal{H}_R$.

A. Existence and Uniqueness of Generalized Current

Existence and uniqueness of scalar currents in infinite networks has been examined in [13], [14]. It was shown by Flanders that, unlike in finite networks, in an infinite electrical network the current is not uniquely determined by Kirchhoff's laws and Ohm's law [13]. He showed, however, that uniqueness of current in an infinite network can be established if two additional conditions are imposed: the current has a finite dissipated energy and it is the limit of flows with finite support. For this reason, in examining the uniqueness of generalized currents in infinite networks we restrict ourselves to generalized flows that are limits of finite support flows and that have finite dissipated energy. For finite networks these conditions hold trivially.

The following theorem establishes existence, uniqueness, and linearity of generalized currents and potential differences in generalized electric networks. The proof of this result is provided in Appendix I.

Theorem 2 (Generalized Current): For every pair of nodes $u, v \in \mathbf{V}$ and intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$, among all flows that have finite dissipated energy and are limits of finite support flows, there exists a unique current i from u to v with intensity $\mathbf{i}$. In addition:

1) the current is the flow that minimizes the energy dissipation, among all flows from node u to node v with intensity $\mathbf{i}$, that are limits of finite support flows;
2) the current i and the potential difference $V_u - V_v$ (for every $u, v \in \mathbf{V}$) are linear functions of the intensity $\mathbf{i}$. The potential is unique only up to an additive constant.

□

It was previously known that in a scalar electrical network, the current minimizes energy dissipation. This result is known as Thomson's Minimum Energy Principle [12], [14]. Theorem 2 shows that generalized currents also obey Thomson's Principle in both finite and infinite networks.

B. Generalized Effective Resistance

It was shown in the previous section that the potential difference $V_u - V_v \in \mathbb{R}^{k\times k}$ associated with a current of intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$ flowing from u to v is a linear function of $\mathbf{i}$. It turns out that this linear map can be expressed through the matrix multiplication by a $k \times k$ matrix, which is stated next. The proof of this result is provided in Appendix I.

Lemma 1: Let $(\mathcal{G}, R)$ be a generalized electric network satisfying Assumption 2 . The linear mapping between $\mathbf{i}$ and $V_u - V_v$ can be defined by multiplication by a $k \times k$ matrix, which we call the *generalized effective resistance* $R^{\text{eff}}_{u,v}$ *between* u *and* v

$$V_u - V_v = R^{\text{eff}}_{u,v}\mathbf{i}, \quad \forall\, \mathbf{i} \in \mathbb{R}^{k\times k}.$$

□

In the sequel, we will refer to generalized effective resistance simply as effective resistance. In view of Lemma 1, the effective resistance between two nodes is the potential difference between them when a current with intensity I_k, the $k \times k$ identity matrix, is injected at one node and extracted at the other, which is analogous to the definition of effective resistance in scalar networks [12]. Moreover, the effective resistance is a symmetric positive-definite matrix. To show this, we will need the following technical result (also proved in Appendix I), which will have additional usefulness in the sequel.

Lemma 2: Let $i \in \mathcal{H}_R$ be the unique current in the network $(\mathcal{G}, R)$ with intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$ from u to v, and let j be a flow with intensity $\mathbf{j} \in \mathbb{R}^{k\times k}$ from u to v that can be expressed as a limit of finite support flows. Then

$$\sum_{e\in\mathbf{E}} i_e^T R_e j_e = (V_u - V_v)^T\mathbf{j}$$

where V is a generalized potential associated with the current i. Moreover, the series in the left-hand side (LHS) converges absolutely, meaning that each one of the k^2 series that constitute the matrix-valued LHS converges absolutely. □

To prove positive-definiteness of effective resistances, set $j = i$ in Lemma 2, where both i and j have intensity I_k, to obtain

$$\sum_{e\in\mathbf{E}} i_e^T R_e i_e = (V_u - V_v)^T = \left(R^{\text{eff}}_{u,v}\right)^T \tag{8}$$

where the second equality follows from the definition of effective resistance in Lemma 1. Since all the generalized edge-resistances R_e are symmetric and positive-definite, we conclude that the LHS must be symmetric and positive-definite, which confirms that effective resistances are indeed symmetric positive-definite.

C. Rayleigh's Monotonicity Law

The next result relates the effective resistances of two distinct networks related by an appropriate partial order. A similar result for finite scalar networks, called Rayleigh's Monotonicity Law [12], states that if the edge-resistances in a scalar electrical network are increased (perhaps even made infinity, i.e., an open circuit), then the effective resistance between every pair of nodes in the network can only increase. For a long time, Rayleigh's Monotonicity Law was considered so evidently true that no proof was deemed necessary. Nevertheless, [12] provided a proof, which we now extend to generalized electrical networks.

Theorem 3 (Generalized Rayleigh's Monotonicity Law): Consider two generalized electrical networks $(\mathcal{G}, R)$ and $(\bar{\mathcal{G}}, \bar{R})$ for which $\mathcal{G} \subset \bar{\mathcal{G}}$ and $R_e \geq \bar{R}_e$ for every $e \in \mathbf{E}$. For every pair of nodes u, v of $\mathcal{G}$,

$$R^{\text{eff}}_{u,v} \geq \bar{R}^{\text{eff}}_{u,v}$$

where $R^{\text{eff}}_{u,v}$ and $\bar{R}^{\text{eff}}_{u,v}$ are the effective resistances between u and v in the networks $(\mathcal{G}, R)$ and $(\bar{\mathcal{G}}, \bar{R})$, respectively. □

Proof of Theorem 3: Let $i : \mathbf{E} \to \mathbb{R}^{k\times k}$ and $\bar{i} : \bar{\mathbf{E}} \to \mathbb{R}^{k\times k}$ be the currents from u to v in the networks $(\mathcal{G}, R)$ and $(\bar{\mathcal{G}}, R)$, respectively, both with intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$. Defining $\bar{j} : \bar{\mathbf{E}} \to \mathbb{R}^{k\times k}$ to be the following "extension" of the current i to the graph $\bar{G}$

$$\bar{j}_e = \begin{cases} i_e, & e \in \mathbf{E} \\ 0, & e \in \bar{\mathbf{E}} \setminus \mathbf{E} \end{cases}$$

we conclude that $\bar{j}$ satisfies the conservation law (5) for the network $(\bar{\mathcal{G}}, R)$ and is therefore a flow for this network (although not necessarily a current). Since according to Theorem 2 the current $\bar{i}$ is the flow of minimum dissipated energy for the network $(\bar{\mathcal{G}}, R)$, we conclude that

$$\begin{aligned} \operatorname{Tr}\left(\sum_{e\in\bar{\mathbf{E}}} \bar{i}_e^T \bar{R}_e \bar{i}_e\right) &\leq \operatorname{Tr}\left(\sum_{e\in\bar{\mathbf{E}}} \bar{j}_e^T \bar{R}_e \bar{j}_e\right) \\ &= \operatorname{Tr}\left(\sum_{e\in\mathbf{E}} i_e^T \bar{R}_e i_e\right) \\ &\leq \operatorname{Tr}\left(\sum_{e\in\mathbf{E}} i_e^T R_e i_e\right) \end{aligned}$$

where the equality is a consequence of the definition of $\bar{j}$ and the last inequality follows from the fact that $\bar{R}_e \leq R_e, \forall\, e \in \mathsf{E}$. From this, Lemma 2, and the definition of effective resistance, we conclude that

$$\operatorname{Tr}\left(\mathbf{i}^T \bar{R}^{\text{eff}}_{u,v}\mathbf{i}\right) \leq \operatorname{Tr}\left(\mathbf{i}^T R^{\text{eff}}_{u,v}\mathbf{i}\right)$$

for every $\mathbf{i} \in \mathbb{R}^{k\times k}$, from which the result follows. ■

D. Approximating Infinite Network Currents

The next theorem shows that currents and effective resistances in an infinite network can be approximated with arbitrary accuracy by those in a sufficiently large but finite subnetwork. A similar result for the usual scalar electrical networks was established by Flanders [13], [14]. The proof of the theorem, which is inspired by [13], is provided in Appendix I.

Theorem 4 (Finite Approximation): Let $(\mathcal{G}, R)$ be a network satisfying Assumption 2, $\{\mathcal{G}^{(n)}\}$ a nested sequence of finite graphs satisfying Assumption 1, and u, v two arbitrary nodes that appear in every graph $\mathcal{G}^{(n)}$. For a given intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$, let i and $i^{(n)}$ denote the currents from node u to node v in the

infinite network $(\mathcal{G}, R)$ and in the finite network $(\mathcal{G}^{(n)}, R)$, respectively. Then

$$\lim_{n\to\infty} i^{(n)} = i$$

where convergence is in the $\mathcal{H}_R$-norm. In addition, denoting by $R^{\text{eff}}_{u,v}$ and $R^{\text{eff}(n)}_{u,v}$ the effective resistances between nodes u and v in the networks $(\mathcal{G}, R)$ and $(\mathcal{G}^{(n)}, R)$, respectively, we have

$$\lim_{n\to\infty} R^{\text{eff}(n)}_{u,v} = R^{\text{eff}}_{u,v}.$$

□

This result will be instrumental in showing that the BLU estimator error covariances in large finite networks converge to the effective resistance in the limiting infinite network.

IV. Electrical Analogy and Proof of Theorem 1

We start by establishing the electrical analogy for finite networks. The proof of Theorem 1 is then provided, which uses these results.

A. BLU Estimation in Finite Networks

The analogy between BLU estimation in a *finite* measurement network and the corresponding electrical network is stated in the next theorem.

Theorem 5 (Finite Electrical Analogy): Let $(\mathcal{G}, P)$ be a measurement network with a finite weakly connected graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ and an edge-covariance function $P : \mathbf{E} \to \mathbb{S}^{k+}$, with node o as the reference node. For every node $u \in \mathbf{V} \setminus \{o\}$, the following statements hold.

1) The BLU estimator C of x_u in the finite measurement network $(\mathcal{G}, P)$ is equal to the current i with identity intensity I_k in the generalized electrical network $(\mathcal{G}, P)$ from u to o.
2) The covariance $\Sigma_{u,o}$ of the BLU estimation error $x_u - \hat{x}_u$ is equal to the effective resistance $R^{\text{eff}}_{u,o}$ between the node u and the reference node o.

□

To prove this theorem, we need the next lemma which shows that in a finite network, an unbiased estimator must be a flow. The proof of the lemma is provided in [15].

Lemma 3 (Unbiased Estimator): In a finite measurement network $(\mathcal{G}, P)$ with a reference node o, an edge function j is a linear unbiased estimator of a node variable x_u if and only if j is a flow of intensity I_k from node u to the reference node o. In this case, the covariance of the error in the estimate $\hat{\hat{x}}_u$ is given by

$$\mathrm{E}\left[(x_u - \hat{\hat{x}}_u)(x_u - \hat{\hat{x}}_u)^T\right] = \sum_{e\in\mathbf{E}} j_e^T P_e j_e.$$

□

Note that we have used $\hat{\hat{x}}_u$ above to distinguish the estimate from the BLU estimate $\hat{x}_u$ of the node variable x_u. The next result provides a necessary and sufficient condition for the existence of linear unbiased estimators in finite networks, whose proof is provided in [15].

Lemma 4: For a finite measurement graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with a reference node $o \in \mathbf{V}$, there exists an unbiased estimator for every node variable x_u, $u \in \mathbf{V} \setminus \{o\}$ if and only if $\mathcal{G}$ is weakly connected. □

The previous result explains the need for the assumption of weak connectivity. We are now ready to prove Theorem 5.

Proof of Theorem 5: From the Unbiased Estimator Lemma 3 and the definition of energy dissipation (7), we see that in a finite network $(\mathcal{G}, P)$ with reference node o, the BLU estimator C of node variable x_u is given by

$$C = \arg\min \|j\| \quad \text{subject to : } j \text{ is a flow of intensity } I_k \text{ from } u \text{ to } o.$$

Comparing with the electrical network problem, we conclude from Theorem 2 that the BLU estimator C of x_u is the current i of intensity I_k from u to o in the generalized electrical network $(\mathcal{G}, P)$, which proves the first statement.

Since $C = i$, it follows from Unbiased Estimator Lemma 3 that the covariance of x_us BLU estimation error is given by

$$\Sigma_{u,o} = \sum_{e\in\mathbf{E}} i_e^T P_e i_e = R^{\text{eff}}_{u,o}$$

where the second inequality follows from (8), which proves the second statement. ■

B. Computation of the Blue in Finite Networks

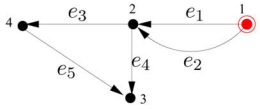
In this section we provide explicit formulas for computing the BLU estimates and the covariances, and remark on distributed computation of the estimates for sensor-network applications. Consider a finite measurement graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ with $|\mathbf{V}| = N$ nodes and $|\mathbf{E}| = M$ edges. Without loss of generality, let $x_o = 0$ and the reference node o be indexed by 1 and the nodes with unknown node variables indexed 2 through N. To express the relationship between the variables and the measurements in a compact form, we use the definition of the *incidence matrix* of a graph from [16]. The incidence matrix A of a directed graph $\mathcal{G}$ consisting of N nodes and M edges is an $N \times M$ matrix, with one row per node and one column per edge. It is defined by $A := [a_{ue}]$, where $a_{ue} = 1$ is if the edge e is incident on the node u and directed away from it, $a_{u,e} = -1$ if e is incident on u but directed toward it, and $a_{u,e} = 0$ if e is not incident on u. We define the *generalized incidence matrix* as

$$\mathcal{A} := A \otimes I_k \in \mathbb{R}^{kN\times kM} \tag{9}$$

where $\otimes$ denotes the Kronecker product. Fig. 2 shows an example of a generalized incidence matrix. Let $\mathbf{x} = \left[x_2^T, x_3^T, \ldots, x_N^T\right]^T \in \mathbb{R}^{k(N-1)}$ the vector of all the unknown node variables. By stacking together all of the measurements into a single vector $\mathbf{z} = \left[z_1^T, z_2^T, \ldots, z_M^T\right]^T \in \mathbb{R}^{kM}$ and all the measurement errors into a vector $\boldsymbol{\epsilon} = \left[\epsilon_1^T, \ldots, \epsilon_M^T\right]^T \in \mathbb{R}^{kM}$, we can express all of the measurement (1) in the compact form

$$\mathbf{z} = \mathcal{A}_b^T \mathbf{x} + \boldsymbol{\epsilon}, \quad \mathcal{A}_b := A_b \otimes I_k \tag{10}$$

where A_b is the *basis incidence matrix* of $\mathcal{G}$, defined as the submatrix of the incidence matrix A obtained after removing the row corresponding to the reference node o. In general, a basis



$$\mathcal{A} = \begin{bmatrix} I_k & I_k & 0 & 0 & 0 \\ -I_k & -I_k & I_k & I_k & 0 \\ 0 & 0 & 0 & -I_k & -I_k \\ 0 & 0 & -I_k & 0 & I_k \end{bmatrix} \qquad \mathcal{A}_b = \begin{bmatrix} -I_k & -I_k & I_k & I_k & 0 \\ 0 & 0 & 0 & -I_k & -I_k \\ 0 & 0 & -I_k & 0 & I_k \end{bmatrix}$$

Fig. 2. A measurement graph $\mathcal{G}$ with node 1 as the reference node, its generalized incidence matrix $\mathcal{A}$, and the generalized basis incidence matrix $\mathcal{A}_b$ constructed with respect to the reference node. The (block) row and (block) column indices of $\mathcal{A}$ correspond to node and edge indices, respectively.

incidence matrix of a directed graph is obtained by removing any row of the incidence matrix [16]. We call the matrix $\mathcal{A}_b$ defined in (10) the *generalized basis incidence matrix* of $\mathcal{G}$. We further construct $\mathscr{P} := \operatorname{diag}(P_1, P_2, \ldots, P_M) \in \mathbb{R}^{kM \times kM}$ as the block diagonal square matrix of all the measurement error covariances. Fig. 2 shows an example of a generalized basis incidence matrix. The next result provides explicit formulas for computing the BLU estimates and error covariances in finite graphs.

Theorem 6: Consider a finite measurement network $(\mathcal{G}, P)$ satisfying Assumption 2 with a graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$. Let the reference node o be indexed by 1, and the vectors $\mathbf{x}$, $\mathbf{z}$, $\boldsymbol{\epsilon}$ and the matrices $\mathcal{A}_b, \mathscr{P}$ be constructed as described above. Then, the BLU estimate of $\mathbf{x}$ is given by

$$\hat{\mathbf{x}} = \mathcal{L}_b^{-1} \mathcal{A}_b \mathscr{P}^{-1} \mathbf{z}, \quad \mathcal{L}_b := \mathcal{A}_b \mathscr{P}^{-1} \mathcal{A}_b^T \tag{11}$$

and the error covariance by

$$\Sigma := \mathrm{E}\left[(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\right] = \mathcal{L}_b^{-1}.$$

□

Proof of Theorem 6: A basis incidence matrix constructed by removing an arbitrary row from the incidence matrix of a graph has full row rank if and only the graph is weakly connected [16]. It follows from Assumption 2 that the generalized basis incidence matrix $\mathcal{A}_b$ of the measurement graph $\mathcal{G}$ also has full row rank. Assumption 2 also ensures that $\mathcal{P} > 0$; hence $\mathcal{L}_b$ is positive-definite. The statements then follow from standard results in least squares estimation applied to the measurement model (10) (see, e.g., [17]). ■

It follows from the theorem above that the BLU covariances, and therefore the generalized effective resistances, can be determined by computing the inverse of $\mathcal{L}_b$. In the special case when $k = 1$ and $\mathscr{P} = I$, $\mathcal{L}_b$ turns out to be the *Dirichlet Laplacian* matrix of the graph $\mathcal{G}$ with the reference node o node as the boundary [18]. For this reason, we call $\mathcal{L}_b$ in (11) the *generalized Dirichlet Laplacian* matrix of the finite network $(\mathcal{G}, P)$. Due to the structure of the matrix $\mathcal{L}_b$, the equation $\mathcal{L}_b \hat{\mathbf{x}} = \mathbf{b}$ is amenable to parallel iterative methods for solving linear equations. Such techniques are used in [7] to devise distributed algorithms to compute the BLU estimate $\hat{\mathbf{x}}$, in which every node computes its own variable's estimate and the information needed to carry out the computation is obtained by communicating with its neighboring nodes.

C. Proof of the Main Result

Theorem 1 can now be proved using the tools developed so far.

Proof of Theorem 1:

We will prove the statements of the theorem in reverse order.

Since the sequence of BLU covariances $\Sigma_{u,o}^{(n)}$ is the same as the sequence of effective resistances $R_{u,o}^{\mathrm{eff}(n)}$ (Finite Electrical Analogy Theorem 5), and the sequence $R_{u,o}^{\mathrm{eff}(n)}$ converges to the effective resistance $R_{u,o}^{\mathrm{eff}}$ in the infinite network (Finite Approximation Theorem 4), we have

$$\Sigma_{u,o}^{(n)} = R_{u,o}^{\mathrm{eff}(n)} \to R_{u,o}^{\mathrm{eff}} =: \Sigma_{u,o}.$$

Moreover, by the construction of the nested sequence $\{\mathcal{G}^{(n)}\}$, if $n_1 \leq n_2$, then $\mathcal{G}^{(n_1)} \subset \mathcal{G}^{(n_2)}$, and so by the Generalized Rayleigh's Monotonicity Law (Theorem 3)

$$\Sigma_{u,o}^{(1)} \geq \Sigma_{u,o}^{(2)} \geq \cdots$$

from which the third statement of the theorem follows.

Moreover, the BLU estimator $C^{(n)}$ of x_u in the finite network $(\mathcal{G}^{(n)}, P)$ is equal to the current $i^{(n)}$ in the generalized electrical network $(\mathcal{G}^{(n)}, P)$ (Finite Electrical Analogy Theorem 5), and the currents $i^{(n)}$ converge to the unique current i in the electrical network $(\mathcal{G}, R)$ (Finite Approximation Theorem 4). Therefore

$$C^{(n)} = i^{(n)} \to i =: C$$

where the convergence is in the $\mathcal{H}_P$-norm. This proves the second statement. By definition of the BLU estimator, we get

$$\begin{aligned} \hat{x}_u^{(n)} &= \sum_{(p,q) \in \mathbf{E}^{(n)}} C_{p,q}^{(n)T} (x_p - x_q + \epsilon_{p,q}) \\ &= x_u + \sum_{(p,q) \in \mathbf{E}^{(n)}} C_{p,q}^{(n)T} \epsilon_{p,q} \end{aligned} \tag{12}$$

where the second equality follows from unbiasedness, since otherwise the expectation of the LHS would not be equal to x_u. Let $n < l$, so that from Assumption 1, $\mathcal{G}^{(n)} \subset \mathcal{G}^{(l)}$. It follows from the uncorrelatedness of the ϵs and (12) that

$$\begin{aligned} &\mathrm{E}\left[\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)^T\right] \\ &\qquad = \sum_{e \in \mathbf{E}^{(l)}} \left(C_e^{(l)} - C_e^{(n)}\right)^T P_e \left(C_e^{(l)} - C_e^{(n)}\right) \end{aligned}$$

where we have used the convention that $C_e^{(n)} = 0$ if $e \in \mathbf{E}^{(l)} \setminus \mathbf{E}^{(n)}$. This leads to

$$\operatorname{Tr}\left(\mathrm{E}\left[\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)^T\right]\right) = \|C^{(l)} - C^{(n)}\|^2$$

where $\|\cdot\|$ is the $\mathcal{H}_P$-norm. Since $C^{(n)} \to i$, $\|C^{(l)} - C^{(n)}\| \to 0$ as $n, m \to \infty$. Therefore

$$\lim_{n\to\infty} \sup_{l\geq n} \mathrm{Tr}\left(\mathrm{E}\left[\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)\left(\hat{x}_u^{(l)} - \hat{x}_u^{(n)}\right)^T\right]\right) = 0. \tag{13}$$

We recall that a sequence of random variables $\{\eta_n\}$ converges in the mean square sense if and only if ([19, proposition 6.3])

$$\lim_{n,l\to\infty} \sup_{l\geq n} \mathrm{E}\left[|\eta_l - \eta_n|^2\right] = 0.$$

Therefore, the sequence of random vectors $\hat{x}_u^{(n)}$ converge entry-wise in the mean square sense. This proves the first statement and completes the proof of Theorem 1. ■

Remark 1 (Role of Edge Directions): Effective resistances are independent of the directions of the edges in the graph. Reversing the direction of an edge e simply reverses the sign of the current i_e on that edge. It follows from (8) that the effective resistance between any two nodes is unaffected by the edge-directions. Therefore, all results in this paper that use the graph partial order defined in Assumption 1 also hold when $\mathcal{G} \subset \bar{\mathcal{G}}$ is understood to mean that the graph $\mathcal{G}$ can be *embedded* in the graph $\bar{\mathcal{G}}$, which means that (1) the nodes of $\mathcal{G}$ can be mapped injectively into the nodes of $\bar{\mathcal{G}}$; and (2) for every edge $e = (u, v)$ in $\mathcal{G}$, there is a corresponding edge $\bar{e}$ in $\bar{\mathcal{G}}$ that is incident on the nodes $\bar{u}$ and $\bar{v}$, where $\bar{u}$ and $\bar{u}$ correspond to u and v, respectively, but edge directions need not be preserved.

It follows from the electrical analogy that, although a measurement graph is directed because of the need to distinguish between a measurement of $x_u - x_v$ and that of $x_v - x_u$, the BLU error covariance is independent of the edge directions. □

V. Blue Covariance in Lattices

In this section, we look at certain special classes of infinite graphs, namely, lattices, for which the effective resistance between two nodes can be analytically derived. Since we can exactly compute the effective resistance, we know the minimum possible variance achievable in these graphs. A numerical study is also presented for lattices to examine the rate at which the BLU estimator variances in a nested sequence of finite subgraphs converge to the minimum possible value as the subgraphs increase in size.

A. Effective Resistance in Lattices

The d-D square lattice $\mathbf{Z}_d$ is defined as a graph with a node in every point in $\mathbb{R}^d$ with integer coordinates and an edge between every pair of nodes at an Euclidean distance equal to one. Edge directions are arbitrary since they play no role in the effective resistance, and therefore in the estimation error covariances (see Remark 1). We construct a generalized electrical network by assigning a constant matrix-resistance to every edge of $\mathbf{Z}_d$.

The following lemma establishes the effective resistance of a d-D square lattice. The *graphical distance* $d_{\mathcal{G}}(u, v)$ between two nodes u and v in a graph $\mathcal{G}$ is the minimum number of edges one has to traverse in order to go from u to v, without necessarily respecting the edge orientations. The graphical distance in the lattice $\mathbf{Z}_d$ is denoted by $d_{\mathbf{Z}_d}(\cdot, \cdot)$.

Lemma 5: Consider the electrical network $(\mathbf{Z}_d, R_o)$ with the same generalized resistance $R_o \in \mathbb{S}^{k+}$ at every edge of the d-D square lattice. The generalized effective resistance $R_{u,v}^{\mathrm{eff}}$ between two nodes u and v in the electrical network $(\mathbf{Z}_d, R_o)$ satisfies

1) $R_{u,v}^{\mathrm{eff}}(\mathbf{Z}_1) = \Theta\left(d_{\mathbf{Z}_1}(u, v)\right)$;
2) $R_{u,v}^{\mathrm{eff}}(\mathbf{Z}_2) = \Theta\left(\log d_{\mathbf{Z}_2}(u, v)\right)$;
3) $R_{u,v}^{\mathrm{eff}}(\mathbf{Z}_3) = \Theta\big(1\big)$.

□

The usual asymptotic notation $\Theta(\cdot)$ is used with matrix valued functions in the following way. For two functions $g : \mathbb{R} \to \mathbb{R}^{k\times k}$ and $f : \mathbb{R} \to \mathbb{R}$, the notation $g(z) = \Theta(f(z))$ means there exists a constant z_o and two matrices $A, B \in \mathbb{S}^{k+}$ that are independent of z such that $Af(z) \leq g(z) \leq Bf(z)$ for all $z > z_o$.

Note that the Generalized Current Theorem 2 guarantees that effective resistances in infinite lattice networks are well defined. The results in Lemma 5 are established by extending known results on the *scalar* effective resistance in the d-D square Lattice to the generalized case.

The next result, whose proof is provided in [15], is needed for the proof of Lemma 5.

Lemma 6: For a given graph $\mathcal{G}$ with finite maximum node degree, let $r_{u,v}^{\mathrm{eff}}$ denote the *scalar* effective resistance between two nodes u and v in an a scalar electrical network $(\mathcal{G}, 1)$ that has 1-Ohm resistors on every edge of the graph $\mathcal{G}$. Let $(\mathcal{G}, R_o)$ be a generalized electrical network constructed from the same graph $\mathcal{G}$ by assigning a generalized resistance $R_o \in \mathbb{S}^{k+}$ to every edge of $\mathcal{G}$. Then

$$R_{u,v}^{\mathrm{eff}} = r_{u,v}^{\mathrm{eff}} R_o.$$

□

Scaling laws for the effective resistances in scalar lattice networks are stated in the next result, which follows from the results established in [20] and [21].

Lemma 7: Consider the electrical network $(\mathbf{Z}_d, 1)$ with the same scalar resistance 1-Ohm at every edge of the d-D square lattice. The scalar effective resistance $r_{u,v}^{\mathrm{eff}}$ between two nodes u and v in the electrical network $(\mathbf{Z}_d, 1)$ satisfies

1) $r_{u,v}^{\mathrm{eff}}(\mathbf{Z}_1) = \Theta\left(d_{\mathbf{Z}_1}(u, v)\right)$;
2) $r_{u,v}^{\mathrm{eff}}(\mathbf{Z}_2) = \Theta\left(\log d_{\mathbf{Z}_2}(u, v)\right)$;
3) $r_{u,v}^{\mathrm{eff}}(\mathbf{Z}_3) = \Theta\big(1\big)$.

□

Lemma 5 now follows from Lemma 7 and Lemma 6.

B. Convergence Rate

Theorem 1 shows that the BLU estimator error variance in a sequence of nested finite subgraphs of an infinite measurement graph converges to a limiting variance that is numerically equal to an effective resistance, regardless of how the sequence $\mathcal{G}^{(n)}$ is constructed. However, the rate at which the covariances $\Sigma_{u,o}^{(n)}$ converge to the effective resistance in the infinite graph will depend on how the sequence $\{\mathcal{G}^{(n)}\}$ is constructed *vis-a-vis* the nodes u and o. One natural way to construct the graph $\mathcal{G}^{(n)} = (\mathbf{V}^{(n)}, \mathbf{E}^{(n)})$ is to take $\mathbf{V}^{(n)}$ to contain all nodes that are at a graphical distance smaller than $\alpha(n)$ from the shortest path connecting u and o, where $\alpha(\cdot)$ is a positive and increasing

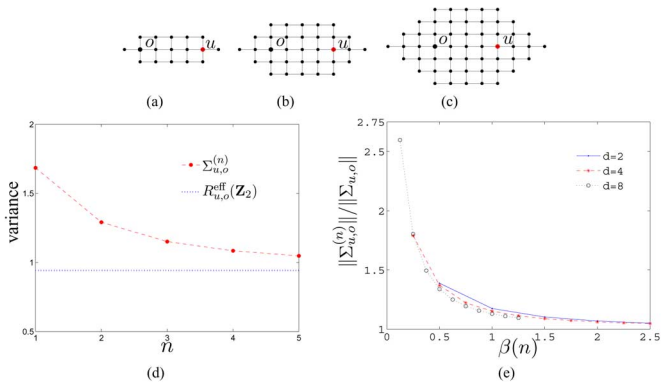


Fig. 3. (a)–(c) The first three members of a sequence of nested subgraphs $\mathbf{Z}_2^{(n)}$ of the 2-D lattice $\mathbf{Z}_2$. (d) The plot of variances $\Sigma_{u,o}^{(n)}$ in the sequence of measurement networks $\left(\mathbf{Z}_2^{(n)}, 1\right)$ as a function of n. The variance in the nested finite subgraphs monotonically decrease toward the limiting value as the subgraphs increase in size. (e) Trend of the ratio of variance in the finite subnetworks $\left(\mathbf{Z}_2^{(n)}, 1\right)$ to the minimum possible variance in $(\mathbf{Z}_2, 1)$, as a function of $\beta(n)$ for three different node pairs u, o, when the node set $\mathbf{V}^{(n)}$ is chosen so as to encompass all nodes within a radius of $\beta(n)d_{u,o}$ from the shortest path connecting u and o.

function. The distance of a node from a path denotes the minimum graphical distance between the node and any node on the path. If there are multiple shortest paths, we take the union of the sets obtained for each shortest path. $\mathbf{E}^{(n)}$ is then chosen as the set of edges that are incident on the nodes in $\mathbf{V}^{(n)}$. This construction satisfies Assumption 1.

Fig. 3(a)–(c) shows the first three members of a sequence of nested subgraphs $\left\{\mathbf{Z}_2^{(n)}\right\}$ of the 2-D lattice $\mathbf{Z}_2$, constructed according to the procedure outlined above, with $\alpha(n) = n$. For simplicity, we consider the case of scalar variables and measurements, and every measurement error is assumed to have a variance 1. Covariances for vector-valued variables could be obtained using Lemma 6. Fig. 3(d) shows the plot of the variances $\Sigma_{u,o}^{(n)}$ of node u in the measurement network $\left(\mathbf{Z}_2^{(n)}, 1\right)$ as a function of n. The limiting value of the variance is the effective resistance between u and o in the infinite lattice $\mathbf{Z}_2$. In an infinite 2-D lattice with unit resistance on every edge, the effective resistance between two nodes u with relative x and y coordinates is given by $R^{\text{eff}}_{u,o} = (1/\pi)\left(\log\sqrt{x^2+y^2} + \gamma + (1/2)\log 8\right)$, where $\gamma \approx 0.577$ [21]. For the example in Fig. 3(a)–(c), $x = 4$, $y = 0$, so the limiting variance for node u is $\Sigma_{u,o} = R^{\text{eff}}_{u,o} \approx 0.956$, which is shown by a dotted line in the Fig. 3(d). As expected, the variances $\Sigma_{u,o}^{(n)}$ monotonically decrease and approach the asymptotic value as n increases.

For a given nested sequence $\mathcal{G}^{(n)}$, the convergence rate of $\Sigma^{(n)}$ to Σ will depend on the graphical distance $d_{u,o}$ between nodes u and o. Taking this into account, we can construct the sequence $\mathcal{G}^{(n)}$ by choosing $\mathcal{V}^{(n)}$ as the set of nodes that are within a graphical distance of $\beta(n)d_{u,o}$ of the shortest path connecting u and o, where $\beta(\cdot)$ is a positive and increasing function. Numerical studies on the 2-D lattice $\mathbf{Z}_2$ indicate that with this construction, the ratio $\left\|\Sigma_{u,o}^{(n)}\right\| / \|\Sigma_{u,o}\|$ depends only on the value of β and is independent of $d_{u,o}$. Fig. 3(e) shows the ratio $\left\|\Sigma_{u,o}^{(n)}\right\| / \|\Sigma_{u,o}\|$ as a function of β for three different nodes taken at distances of 2, 4, and 8, respectively, from o. The figure shows that the rate of convergence of $\Sigma_{u,o}^{(n)}$ to the limiting value $\Sigma_{u,o}$ is not sensitive to the distance between u and o. In particular, with $\beta = 2$, the error between $\Sigma^{(n)}$ and Σ is less than 10%. These studies show that in a 2-D lattice, a relatively small subgraph is sufficient to obtain an estimate whose variance is quite close to the minimum possible achievable by using all the measurements. For an arbitrary measurement graph, as long as the graph is "close to" a lattice in an appropriate sense, similar trends are expected. For details on appropriate measures of closeness to lattices, the reader is referred to [11].

VI. Summary

The problem of estimating vector-valued node variables from noisy relative measurements naturally arises in many applications that have a graphical structure, such as localization in sensor networks and motion consensus in swarms of mobile agents. In this paper, we obtained a characterization on the covariance of the minimum possible estimation error when an arbitrarily large number of measurements is available. This covariance was shown to be equal to a matrix-valued effective resistance in an infinite electrical network. We also showed that when the measurement graph has bounded node degree, the error covariance of the estimate produced by using only a finite subset of measurements converges to the error covariance of the estimate that could be obtained by using all the available measurements.

The convergence results established in this paper provide the formal justification for treating large finite graphs as infinite graphs, which can be used to obtain asymptotic bounds on the estimation error. Preliminary work on determining such asymptotic error bounds for infinite graphs has been reported in [11]. Further work on establishing scaling laws using the tools developed in this paper is under way. Furthermore, the matrix-valued effective resistance is also relevant in several other distributed control problems [6]. The tools developed in the paper are therefore useful in answering scalability questions in certain distributed control problems as well.

Another implication of the convergence results established in this paper is that for estimation problems based on relative measurements, after a certain point, considering more measurements will only marginally improve the quality of the estimate. This observation may be of interest to designers of distributed estimation algorithms, since it shows that accurate estimation is possible by considering a relatively small subset of measurements among all the available ones. Developing such algorithms, and methods to choose the best possible subset of measurements, are topics for future research.

Our problem formulation assumes zero-mean measurement noise. If this is not the case, BLU estimators produce biased estimates. In fact, it may happen that the bias increases as the number of measurements increases, which could provide additional justification for using only a small subset of the available measurements. A detailed study of the impact of measurement-noise bias remains a problem for future research.

APPENDIX
PROOFS

We first introduce some terminology. Define a norm for all node-functions $\omega : \mathbf{V} \to \mathbb{R}^{k\times k}$ as

$$\|\omega\| = \left(\sum_{u\in\mathbf{V}} \mathrm{Tr}\left(\omega_u^T\omega_u\right)\right)^{(1/2)} = \left(\sum_{u\in\mathbf{V}} \|\omega_u\|_F^2\right)^{(1/2)} \quad (14)$$

where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, and a linear vector space $S_{\mathbf{V}}$ as the space of all bounded node-functions with respect to the above defined norm:

$$S_{\mathbf{V}} = \{\omega : \mathbf{V} \to \mathbb{R}^{k\times k} | \ \|\omega\| < \infty\}. \quad (15)$$

For an infinite network $(\mathcal{G}, R)$, we introduce the *incidence operator* $\mathcal{A} : \mathcal{H}_R \to S_{\mathbf{V}}$, which is defined by the transformation:

$$(\mathcal{A}j)_u = \sum_{e\in\mathbf{E}} a_{u,e} j_e, \quad j \in \mathcal{H}_R \quad (16)$$

where $a_{u,e}$ is nonzero if and only if the edge e is incident on the node u and when nonzero, $a_{u,e} = -1$ if the edge e is directed towards u and $a_{u,e} = 1$ otherwise. The incidence operator $\mathcal{A}$ is simply an extension to infinite graphs of the generalized incidence matrix defined in Section IV-A [see(9)] for finite graphs. The series in(16) is absolutely convergent since it involves only a finite number of terms due to the bounded degree of $\mathcal{G}$.

We call a node-function $\omega \in S_{\mathbf{V}}$ a *divergence* for the graph $\mathcal{G}$ if ω has finite support and $\sum_{u\in\mathbf{V}} \omega_u = 0$. One can view a divergence as an assignment of flow sources at a finite number of nodes of the graph so that total flow into the graph is equal to the total flow out of it.

An edge-function $j \in \mathcal{H}_R$ is called a *flow in $\mathcal{G}$ with divergence* $\omega \in S_{\mathbf{V}}$ if ω is a divergence in $\mathcal{G}$ and j satisfies

$$\sum_{\substack{(u,v)\in\mathbf{E}\\ u=\bar{u}}} j_{u,v} - \sum_{\substack{(v,u)\in\mathbf{E}\\ u=\bar{u}}} j_{v,u} = \omega_{\bar{u}}, \quad \forall\, \bar{u} \in \mathbf{V}. \quad (17)$$

Equation (17) can be compactly represented as

$$\mathcal{A}j = \omega. \quad (18)$$

An edge-function $j \in \mathcal{H}_R$ is called a *circulation in* $(\mathcal{G}, R)$ if

$$\mathcal{A}j = 0. \quad (19)$$

In other words, a circulation is an element of $\mathcal{H}_R$ that belongs to $\mathcal{N}(\mathcal{A})$, the null space of $\mathcal{A}$.

First we show that the linear operator $\mathcal{A} : \mathcal{H}_R \to S_{\mathbf{V}}$ defined above is bounded. Since for each $u \in \mathbf{V}$, $(\mathcal{A}j)_u \in \mathbb{R}^{k\times k}$, we have

$$\|(\mathcal{A}j)_u\|_F^2 = \left\|\sum_{e\in\mathbf{E}_u} a_{ue} j_e\right\|_F^2 \le \sum_{e\in\mathbf{E}_u} \|j_e\|_F^2$$

where $\mathbf{E}_u$ is the set edges in $\mathbf{E}$ that are incident on u. It can be shown from the relationship between the Frobenius norm and the singular values of a matrix that for every edge $e \in \mathbf{E}$, we have $\|j_e\|_F^2 \le (1/\lambda_{\min})\mathrm{Tr}\left(j_e^T R_e j_e\right)$, where $\lambda_{\min}$ is the uniform lower bound on the smallest eigenvalue of $R_e, \forall e \in \mathbf{E}$. Existence of a positive $\lambda_{\min}$ is guaranteed by Assumption 2 . Since the above is true for every $u \in \mathbf{V}$, from (14) we get

$$\begin{aligned}\|\mathcal{A}j\|^2 &= \sum_{u\in\mathbf{V}} \|(\mathcal{A}j)_u\|_F^2 \\ &\le \frac{1}{\lambda_{\min}} \sum_{u\in\mathbf{V}} \sum_{e\in\mathbf{E}_u} \mathrm{Tr}\left(j_e^T R_e j_e\right) \\ &\le \frac{d_{\max}}{\lambda_{\min}} \sum_{e\in\mathbf{E}} \mathrm{Tr}\left(j_e^T R_e j_e\right) = \frac{d_{\max}}{\lambda_{\min}} \|j\|^2\end{aligned}$$

where $d_{\max}$ is the largest degree of the nodes of the graph $\mathcal{G}$, which is finite by Assumption 2. It follows that:

$$\|\mathcal{A}\| \le \sqrt{\frac{d_{\max}}{\lambda_{\min}}}$$

which shows that $\mathscr{A}$ is bounded.

Now we are ready to prove the Generalized Current Theorem 2.

Proof of Theorem 2: We first prove that among the flows in $\mathcal{H}_R$ that are limits of finite support flows, the flow with the minimum dissipated energy exists and is unique, and that this flow is a current. Then we show that there can be only one such current.

For a flow of intensity $\mathbf{j}$ that is injected at u and extracted at v, the corresponding divergence $\bar{\omega}$ is given by $\bar{\omega}_u = \mathbf{j}$, $\bar{\omega}_v = -\mathbf{j}$ and $\bar{\omega}_p = 0$ for all $p \in \mathbf{V} \setminus \{u, v\}$. Pick a path $\mathcal{P}$ from u to v, and construct a flow j^{path} of intensity $\mathbf{j}$ from u to v along $\mathcal{P}$ as follows:

$$j_e^{\mathrm{path}} = \begin{cases} \mathbf{j}, & e \in \mathcal{P}, \vec{e} = \vec{\mathcal{P}} \\ -\mathbf{j}, & e \in \mathcal{P}, \vec{e} \ne \vec{\mathcal{P}} \\ 0, & e \notin \mathcal{P}. \end{cases}$$

It is easy to see that j is a finite support edge-function in $\mathcal{H}_R$ that satisfies the constraint equation $\mathcal{A}j = \bar{\omega}$. All flows satisfying this constraint lie in the linear variety $j^{\mathrm{path}} + \mathcal{N}(\mathcal{A})$, where $\mathcal{N}(\mathcal{A})$ is the null space of $\mathscr{A}$. Since $\mathscr{A}$ is a bounded linear operator, its null space is closed. As a result, $\mathcal{N}(\mathcal{A})$, which is the space of all circulations, is a Hilbert space. Consider the subspace of $\mathcal{N}(\mathcal{A})$ that consists of all finite support circulations, and denote it by $\mathcal{N}_F(\mathcal{A})$ ("F" for finite support). Its closure $\overline{\mathcal{N}_F(\mathcal{A})}$ is a closed subspace of the Hilbert space $\mathcal{N}(\mathcal{A})$. By the Projection Theorem applied to linear varieties ([22, Theorem 1 in section 3.10]), there exists a unique edge-function in $j^{\mathrm{path}} + \overline{\mathcal{N}_F(\mathcal{A})}$ of minimum norm, which we call i, and which is orthogonal to $\overline{\mathcal{N}_F(\mathcal{A})}$.

Since $i - j^{\mathrm{path}} \in \overline{\mathcal{N}_F(\mathcal{A})}$, there exists a sequence of finite support circulations $c^{(n)}$ such that $c^{(n)} \to (i - j^{\mathrm{path}})$, where the convergence is in $\mathcal{H}_R$ norm. Define $j^{(n)} := j^{\mathrm{path}} + c^{(n)}$, so that by construction, each $j^{(n)}$ is a finite support flow of intensity $\mathbf{i}$ from u to v, and $j^{(n)} \to i$ in $\mathcal{H}_R$. This establishes the existence and uniqueness of the flow with minimum power dissipation that is the limit of a sequence of finite support flows.

Since i is orthogonal to $\overline{\mathcal{N}_F(\mathcal{A})}$

$$\langle i, c\rangle = 0 \quad (20)$$

for every $c \in \overline{\mathcal{N}_F(\mathcal{A})}$. Declare the generalized potential drop across an edge e as $R_e i_e$ to satisfy Ohm's law. If the graph has no loops, Kirchhoff's loop law is trivially satisfied by these generalized potential drops. If the graph has loops, pick a loop $\mathcal{C}$ and define a scalar edge-function $f : \mathbf{E} \to \mathbb{R}$ as

$$f_e = \begin{cases} 1, & \text{if } e \in \mathcal{C} \text{ and } \vec{e} = \vec{\mathcal{C}}, \\ -1, & \text{if } e \in \mathcal{C} \text{ and } \vec{e} \neq \vec{\mathcal{C}}, \\ 0, & \text{if } e \notin \mathcal{C}. \end{cases}$$

Now define a finite support circulation c^* as $c^*_e = f_e J$, where J is an arbitrary $k \times k$ matrix. We have

$$\begin{aligned} 0 = \langle i, c^* \rangle &= \sum_{e \in \mathcal{C}} \mathrm{Tr}\left(i_e^T R_e c_e^*\right) = \sum_{e \in \mathcal{C}} f_e \mathrm{Tr}\left(i_e^T R_e J\right) \\ &= \sum_{e \in \mathcal{C}} f_e \mathrm{Tr}\left(J^T R_e i_e\right) \\ &= \mathrm{Tr}\left[J^T \left(\sum_{e \in \mathcal{C}} f_e R_e i_e\right)\right]. \end{aligned}$$

Since this is true for arbitrary J, we must have

$$\sum_{e \in \mathcal{C}} [f_e (R_e i_e)] = 0 \tag{21}$$

which in turn must be true for every loop $\mathcal{C}$, since the arguments above can be repeated for every loop. Equation (21), therefore, shows that the *net potential drop* along every loop is 0. In other words, the generalized potential drops determined by i in accordance with Ohm's law satisfies Kirchhoff's loop law. Construction of a generalized node potential function V is now trivial. Therefore, i is a generalized current.

To prove uniqueness of the current, let i and $\bar{i}$ be two currents from u to v with intensity $\mathbf{i}$. Define an edge-function $d : \mathbf{E} \to \mathbb{R}^{k \times k}$ as $d_e := i_e - \bar{i}_e$. We see that $d \in \overline{\mathcal{N}_F(\mathcal{A})}$. From linearity of the inner product

$$\langle d, d \rangle = \langle i - \bar{i}, i - \bar{i} \rangle = \langle i, d \rangle - \langle \bar{i}, d \rangle = 0 - 0$$

where the last equalities follows from (20), since by construction, both i and $\bar{i}$ are currents. Thus,

$$\sum_{e \in \mathsf{E}} \mathrm{Tr}\left(d_e^T R_e d_e\right) = 0 \quad \Rightarrow d_e = 0 \quad \forall\, e \in \mathbf{E},$$

since $R_e > 0$ for all edges $e \in \mathbf{E}$. We, therefore, conclude that $i = \bar{i}$, which proves that the current i is unique.

To examine the uniqueness of potentials, suppose that V and $\bar{V}$ are two potentials associated with the same current. Because of Ohm's Law, we conclude that

$$V_u - V_v = \bar{V}_u - \bar{V}_v \Rightarrow D_u = D_v, \quad \forall\, (u, v) \in \mathbf{E}$$

where $D = V - \bar{V}$. Since $\mathcal{G}$ is connected, D must be a constant, but is otherwise arbitrary. This shows that the node potentials are unique up to an additive constant.

If i is a current with intensity $\mathbf{i}$ and $\bar{i}$ is a current with intensity $\bar{\mathbf{i}}$, both from u to v, it can be shown in a straightforward manner that $\alpha i + \beta \bar{i}$ is also a current with intensity $\alpha \mathbf{i} + \beta \bar{\mathbf{i}}$ from u to v, from which the linearity from $\mathbf{i}$ to i follows. A similar linearity proof also holds for the potential differences. ■

The corollary presented next is essentially a repetition of (20), but is restated because of its usefulness in several subsequent proofs.

Corollary 1: A flow i is the generalized current in the network $(\mathcal{G}, R)$ if and only if

$$\langle i, c \rangle = 0$$

for every circulation $c \in \overline{\mathcal{N}_F(\mathcal{A})}$. □

Proof of Lemma 1: For the current with intensity $\mathbf{i}$ flowing from u to v, we define a divergence ω as

$$\omega_p = 0 \quad \forall\, p \in \mathbf{V} \setminus \{u, v\}, \omega_u = \mathbf{i}, \omega_v = -\mathbf{i}.$$

The flow constraint now becomes $\mathcal{A} j = \omega$. The current i is the flow that satisfies this constraint and minimizes the energy dissipation $\sum_{e \in \mathbf{E}} \mathrm{Tr}\left(j_e^T R_e j_e\right)$, as shown in Theorem 2 . For every node $p \in \mathbf{V}$, the flow constraint becomes

$$(\mathcal{A} j)_p = \omega_p \Rightarrow \sum_{e \in \mathbf{E}_p} a_{p,e} j_e = \omega_p. \tag{22}$$

Recognizing that this is a $k \times k$ matrix equation, we express it as k separate vector equations:

$$\sum_{e \in \mathbf{E}_p} a_{p,e} j_{e,l} = \omega_{p,l}, \quad l = 1, \ldots, k$$

where the second subscript l represents the lth column of the corresponding matrix. It is easy to see that, for every l, the constraints on the lth column of j_es depend only on the lth column of ω_p, and therefore on the lth column of $\mathbf{i}$. As a result, the solution to this optimization problem is equivalent to solving k separate problems "minimize $\sum_{e \in \mathsf{E}} j_{e,l}^T R_e j_{e,l}$ subject to $\mathcal{A} j_l = \omega_l$," for $l = 1, \ldots, k$, where the edge function j_l and the node function ω_l are now vector-valued: $j_l : \mathbf{E} \to \mathbb{R}^k$, $\omega_l : \mathbf{V} \to \mathbb{R}^k$, the spaces $\mathcal{H}_R$ and $S_{\mathbf{V}}$ are appropriately redefined, and the incidence operator $\mathcal{A}$ has the same definition as in (16) with respect to the new spaces $\mathcal{H}_R$, $S_{\mathbf{V}}$. Because of column-wise independence of the current on the intensities, the matrix current on every edge is obtained by stacking the k vector-valued currents on that edge as columns. For every vector-valued current intensity $\mathbf{i}_l$, $l = 1, \ldots, k$, we obtain a corresponding vector-valued potential difference $V_{u,l} - V_{v,l}$. Again, the matrix-valued potential difference $V_u - V_v$ resulting from the original problem consists of the k columns that are the vector-valued potential difference $V_{u,l} - V_{v,l}$ resulting from the k separate optimization problems described above.

These k separate optimization problems can be solved to determine the vector-valued edge currents in the same manner that the single optimization problem was solved in the proof of Theorem 2 to determine the matrix valued edge currents. In fact, only one of these k problems needs to be solved. To understand why, we first note that the linearity between the matrix valued quantities $\mathbf{i}$ and $V_u - V_v$ that was established in Theorem 2 will be retained between the corresponding vector-valued quantities. Specifically, when a vector-valued current i_l flows from u to v with vector intensity $\mathbf{i}_l$, the vector-valued voltage drop

$V_{u,l} - V_{v,l}$ will be a linear function of the vector intensity $\mathbf{i}_l$, which will be in general a $k \times k$ matrix. Let $R^{\text{eff}}_{u,v} \in \mathbb{R}^{k\times k}$ be this matrix. Then

$$V_{u,l} - V_{v,l} = R^{\text{eff}}_{u,v}\mathbf{i}_l, \quad \forall\, \mathbf{i}_l \in \mathbb{R}^k. \tag{23}$$

From linearity, the same is true for every $l = 1, \ldots, k$. Stacking together the k columns in (23), for $l = 1, \ldots, k$, we get $V_u - V_v = R^{\text{eff}}_{u,v}\mathbf{i}$, which proves that the linear mapping between matrix intensity $\mathbf{i}$ and matrix-valued potential drop $V_u - V_v$ is the $k \times k$ matrix $R^{\text{eff}}_{u,v}$. ∎

Proof of Lemma 2: Pick a path $\mathcal{P}$ from u to v, and construct a flow j^{path} of intensity $\mathbf{j}$ from u to v along $\mathcal{P}$ as follows:

$$j^{\text{path}}_e = \begin{cases} \mathbf{j}, & e \in \mathcal{P}, \vec{e} = \vec{\mathcal{P}} \\ -\mathbf{j}, & e \in \mathcal{P}, \vec{e} \neq \vec{\mathcal{P}} \\ 0, & e \notin \mathcal{P}. \end{cases}$$

The assumed properties of j imply that $j \in j^{\text{path}} + \overline{\mathcal{N}_F(\mathcal{A})}$. Let $j^{(n)}$ be a sequence of finite support flows in $(\mathcal{G}, R)$ that converge to the flow j, i.e., $j^{(n)} \to j$ in $\mathcal{H}_R$. Define

$$c := j - j^{\text{path}}$$
$$c^{(n)} := j^{(n)} - j^{\text{path}}.$$

The function $c \in \mathcal{H}_R$ is a circulation since it is the difference between two flows of the same intensity between the same two nodes. Moreover, $\{c^{(n)}\}$ is a sequence of finite-support circulations that converge to c in $\mathcal{H}_R$. Now, since $c^{(n)}$ is a finite support circulation, from Corollary 1, $\langle i, c^{(n)}\rangle = \langle i, j^{\text{path}} - j^{(n)}\rangle = 0$ for every n, and, therefore

$$\lim_{n\to\infty} \langle i, j^{\text{path}} - j^{(n)}\rangle = 0.$$

Using linearity and continuity of the inner product, we therefore conclude that

$$\begin{aligned} \lim_{n\to\infty} \langle i, j^{(n)}\rangle = \langle i, j^{\text{path}}\rangle \Rightarrow \langle i, j\rangle = \langle i, j^{\text{path}}\rangle \\ \Rightarrow \sum_{e\in\mathbf{E}} \operatorname{Tr}\left(i_e^T R_e j_e\right) = \sum_{e\in\mathcal{P}} \operatorname{Tr}\left((R_e i_e)^T j_e^{\text{path}}\right) \\ = \operatorname{Tr}\left((V_u - V_v)^T \mathbf{j}\right). \end{aligned} \tag{24}$$

Since $i, j \in \mathcal{H}_R$, denoting the sth column of i_e by $i_{s,e}$ and the tth column of j_e by $j_{t,e}$, we can show from (24) using straightforward algebraic manipulation that

$$q_{s,t} := \sum_{e=1}^{\infty} i_{s,e}^T R_e j_{t,e} < \infty. \quad \forall\, s, t = \{1, \ldots, k\}$$

and that the series converges absolutely for every s and t. Define the matrix Q by $[Q]_{s,t} = q_{s,t}$. Since the series converges, for every $\epsilon > 0$, we can choose N large enough such that

$$\left\| \sum_{e=1}^{N} i_e^T R_e j_e - Q \right\| < \epsilon$$

where $\|\cdot\|$ represents any matrix norm. We thus conclude that since $i, j \in \mathcal{H}_R$, the series $\sum_{e\in\mathbf{E}} i_e^T R_e j_e$ converges absolutely to a $k \times k$ matrix. Since (24) holds for an arbitrary $\mathbf{j}$, it can be shown in a straightforward manner that the series $\sum_{e\in\mathbf{E}} i_e^T R_e j_e$ must converge to $(V_u - V_v)^T\mathbf{j}$. Therefore, we get the desired result

$$\sum_{e\in\mathbf{E}} i_e^T R_e j_e = (V_u - V_v)^T \mathbf{j}.$$

∎

Proof of Theorem 4: For every $\epsilon > 0$, we can find a finite-support flow $j^{(n)}$ from u to v of intensity $\mathbf{i}$ such that

$$\|i - j^{(n)}\| < \epsilon \tag{25}$$

which follows from the characterization of the current i in Theorem 2. Pick a finite subgraph $\mathcal{G}^{(n)} = (\mathbf{V}^{(n)}, \mathbf{E}^{(n)})$ of $\mathcal{G}$ from the nested sequence $\{\mathcal{G}^{(n)}\}$ such that the support of $j^{(n)}$ lies in $\mathcal{G}^{(n)}$ (i.e., the edges on which $j^{(n)}$ is not zero are in $\mathbf{E}^{(n)}$). Note that by construction $u, v \in \mathbf{V}^{(n)}$. Denoting by $i^{(n)}$ the current in $(\mathcal{G}^{(n)}, R)$, it follows from Corollary 1 that for a circulation $c^{(n)}$ whose support lies in $\mathcal{G}^{(n)}$:

$$\begin{aligned} \langle i^{(n)}, c^{(n)}\rangle = 0, \text{and } \langle i, c^{(n)}\rangle = 0. \\ \Rightarrow \left|\langle i - j^{(n)}, c^{(n)}\rangle\right| = \left|\langle i, c^{(n)}\rangle - \langle j^{(n)}, c^{(n)}\rangle\right| \\ = \left|\langle i^{(n)}, c^{(n)}\rangle - \langle j^{(n)}, c^{(n)}\rangle\right| \\ = \left|\langle i^{(n)} - j^{(n)}, c^{(n)}\rangle\right|. \end{aligned}$$

Pick $c^{(n)} = i^{(n)} - j^{(n)}$, which, being a difference of two finite support flows from u to v with the same intensity, is a finite support circulation. Furthermore, its support lies in $\mathcal{G}^{(n)}$ since both $i^{(n)}$ and $j^{(n)}$ have their support in $\mathcal{G}^{(n)}$. For this choice of $c^{(n)}$ in the equation above, we get

$$\begin{aligned} \left|\langle i - j^{(n)}, i^{(n)} - j^{(n)}\rangle\right| = \|i^{(n)} - j^{(n)}\|^2 \\ \Rightarrow \|i^{(n)} - j^{(n)}\|^2 \le \|i - j^{(n)}\|\|i^{(n)} - j^{(n)}\| \end{aligned}$$

from the Cauchy Schwarz inequality. Therefore

$$\|i^{(n)} - j^{(n)}\| \le \|i - j^{(n)}\| < \epsilon$$

from (25). From the triangle inequality, we now get

$$\|i - i^{(n)}\| \le \|i - j^{(n)}\| + \|i^{(n)} - j^{(n)}\| < 2\epsilon$$

which proves the statement that $i^{(n)} \to i$ in $\mathcal{H}_R$.

To prove the convergence of the effective resistances, pick an arbitrary $\mathbf{i} \in \mathbb{R}^{k\times k}$ and let i and $i^{(n)}$ be the currents with intensity $\mathbf{i}$ from u to v in $(\mathcal{G}, R)$ and $(\mathcal{G}^{(n)}, R)$, respectively. Lemma 2 implies

$$\begin{aligned} \sum_{e\in\mathbf{E}} i_e^T R_e i_e = \mathbf{i}^T R^{\text{eff}}_{u,v}\mathbf{i}, \\ \sum_{e\in\mathbf{E}} i_e^{(n)T} R_e i_e^{(n)} = \sum_{e\in\mathbf{E}^{(n)}} i_e^{(n)T} R_e i_e^{(n)} = \mathbf{i}^T R^{\text{eff}(n)}_{u,v}\mathbf{i}, \\ \sum_{e\in\mathbf{E}} i_e^T R_e i_e^{(n)} = \mathbf{i}^T R^{\text{eff}}_{u,v}\mathbf{i} \end{aligned}$$

where the last equality uses the fact that $i^{(n)}$ is a flow in $\mathcal{G}$ with intensity $\mathbf{i}$ (though not a current). Therefore

$$\sum_{e\in\mathbf{E}} \mathrm{Tr}\left(\left(i-i_e^{(n)}\right)^T R_e\left(i_e-i_e^{(n)}\right)\right) = \mathrm{Tr}\left(\mathbf{i}^T\left(R^{\mathrm{eff}}(n)_{u,v}-R^{\mathrm{eff}}_{u,v}\right)\mathbf{i}\right).$$

Since $i \to i^{(n)}$ in $\mathcal{H}_R$, the left-hand side (LHS) goes to 0 as $n\to\infty$. Since this is true for arbitrary $\mathbf{i}$, $R^{\mathrm{eff}(n)}_{u,v} \to R^{\mathrm{eff}}_{u,v}$. ■

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REFERENCES

[1] A. Basu, J. Gao, J. Mitchell, and G. Sabhnani, "Distributed localization by noisy distance and angle information," in *Proc. 7th ACM Int. Symp. Mobile Ad Hoc Networking and Computing (MobiHoc)*, 2006, pp. 262–273.

[2] P. Barooah, N. M. Da Silva, and J. P. Hespanha, "Distributed optimal estimation from relative measurements for localization and time synchronization," in *Distributed Computing in Sensor Systems*, ser. Lecture Notes in Computer Science, P. B. Gibbons, T. Abdelzaher, J. Aspnes, and R. Rao, Eds. New York: Springer, 2006, vol. 4026, pp. 266–281.

[3] R. Moses and R. Patterson, "Self-calibration of sensor networks," *SPIE: Unattended Ground Sensor Technologies and Applications IV*, vol. 4743, pp. 108–119, 2002.

[4] R. Karp, J. Elson, D. Estrin, and S. Shenker, "Optimal and global time synchronization in sensornets," Center for Embedded Networked Sensing, Univ. of Calif., Los Angeles, CA, 2003.

[5] R. Solis, V. S. Borkar, and P. R. Kumar, "A new distributed time synchronization protocol for multihop wireless networks," in *Proc. 45th IEEE Conf. Decision Control*, 2006.

[6] P. Barooah and J. P. Hespanha, "Graph effective resistances and distributed control: Spectral properties and applications," in *Proc. 45th IEEE Conf. Decision Control*, 2006, pp. 3479–3485.

[7] P. Barooah and J. P. Hespanha, "Estimation from relative measurements: Algorithms and scaling laws," *IEEE Contr. Syst. Mag.*, vol. 27, no. 4, pp. 57–74, 2007a.

[8] K. Whitehouse and D. Culler, "Calibration as parameter estimation in sensor networks," in *Proc. 1st ACM Int. Wkshp. Wireless Sensor Networks Applicat.*, 2002, pp. 59–67.

[9] N. Patwari, A. O. Hero, III, and M. Perkins, "Relative location estimation in wireless sensor networks," *IEEE Trans. Signal Process.*, vol. 51, no. 8, pp. 2137–2148, Aug. 2003.

[10] D. Moore, J. Leonard, D. Rus, and S. Teller, "Robust distributed network localization with noisy range measurements," in *Proc. 2nd ACM Conf. Embedded Netw. Sens. Syst.*, 2004, pp. 50–61.

[11] P. Barooah and J. P. Hespanha, "Estimation from relative measurements: Error bounds from electrical analogy," in *Proc. 2nd Int. Conf. Intell. Sens. Inf. Process. (ICISIP)*, 2005, pp. 88–93.

[12] P. G. Doyle and J. L. Snell, *Random Walks and Electric Networks*. New York: Math. Assoc. of Amer., 1984.

[13] H. Flanders, "Infinite networks: I—Resistive networks," *IEEE Trans. Circuit Theory*, vol. CT—18, no. 3, pp. 326–331, 1971.

[14] A. H. Zemanian, *Infinite Electrical Networks*. Providence, RI: Cambridge Tracts in Mathematics. Cambridge Univ. Press, 1991.

[15] P. Barooah and J. P. Hespanha, Estimation from relative measurements: electrical analogy and large graphs Center for Control, Dynamical Systems and Computation, Univ. Calif., Santa Barbara, 2007 [Online]. Available: http://ccdc.mee.ucsb.edu/Year/2007.html

[16] W.-K. Chen, *Applied Graph Theory*. Amsterdam, The Netherlands: North-Holland, 1971.

[17] J. M. Mendel, *Lessons in Estimation Theory for Signal Processing, Communications and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1995.

[18] F. Chung, "Spectral graph theory," in *Regional Conf. Series in Math.*, Providence, RI, 1997.

[19] E. Wong, *Stochastic Processes in Information and Dynamical Systems*. Huntington, NY: Robert E. Krieger, 1979.

[20] D. Atkinson and F. v. Steenwijk, "Infinite resistive lattices," *Amer. J. Phys.*, vol. 67, pp. 486–492, 1999.

[21] J. Cserti, "Application of the lattice green's function for calculating the resistance of and infinite network of resistors," *Amer. J. Phys.*, vol. 68, no. 10, pp. 896–906, 2000.

[22] D. G. Luenberger, *Optimization by Vector Space Methods*. New York: Wiley, 1969.

Prabir Barooah (M'05) received the B.Tech. and M.S. degrees in mechanical engineering from the Indian Institute of Technology, Kanpur, in 1996 and the University of Delaware, Newark, in 1999, respectively. He received the Ph.D. degree in electrical and computer engineering from the University of California, Santa Barbara, in 2007.

From 1999 to 2002, he was an Associate Research Engineer with United Technologies Research Center. His research interests include system identification, decentralized and cooperative control of multiagent systems, and estimation in large-scale sensor networks. He is currently an Assistant Professor of Mechanical and Aerospace Engineering with the University of Florida, Gainesville.

João P. Hespanha (M'94–F'08) was born in Coimbra, Portugal, in 1968. He received the Licenciatura in electrical and computer engineering from the Instituto Superior Técnico, Lisbon, Portugal, in 1991 and the Ph.D. degree in electrical engineering and applied science from Yale University, New Haven, CT, in 1998, respectively.

He is currently a Professor with the Department of Electrical and Computer Engineering, University of California, Santa Barbara. From 1999 to 2001, he was an Assistant Professor with the University of Southern California, Los Angeles. His research interests include hybrid and switched systems; the modeling and control of communication networks; distributed control over communication networks (also known as networked control systems); the use of vision in feedback control; stochastic modeling in biology; and the control of haptic devices. He is the author of numerous technical papers and the PI and co-PI in several federally funded projects.

Dr. Hespanha is the recipient of the Yale University's Henry Prentiss Becton Graduate Prize for exceptional achievement in research in Engineering and Applied Science, a National Science Foundation CAREER Award, the 2005 best paper award at the 2nd International Conference on Intelligent Sensing and Information Processing, the 2005 Automatica Theory/Methodology best paper prize, and the 2006 George S. Axelby Outstanding Paper Award. Since 2003, he has been an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.