Error Scaling Laws for Linear Optimal Estimation From Relative Measurements

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***Abstract*—In this paper, we study the problem of estimating vector-valued variables from noisy "relative" measurements, which arises in sensor network applications. The problem can be posed in terms of a graph, whose nodes correspond to variables and edges to noisy measurements of the difference between two variables. The optimal (minimum variance) linear unbiased estimate of the node variables, with an arbitrary variable as the reference, is considered. This paper investigates how the variance of the estimation error of a node variable grows with the distance of the node to the reference node. A classification of graphs, namely, dense or sparse in $\mathbb{R}^d, 1 \leq d \leq 3$, is established that determines this growth rate. In particular, if a graph is dense in 1-D, 2-D, or 3-D, a node variable's estimation error is upper bounded by a linear, logarithmic, or bounded function of distance from the reference. Corresponding lower bounds are obtained if the graph is sparse in 1-D, 2-D, and 3-D. These results show that naive measures of graph density, such as node degree, are inadequate predictors of the estimation error. Being true for the optimal linear unbiased estimate, these scaling laws determine algorithm-independent limits on the estimation accuracy achievable in large graphs.**

***Index Terms*—Covariance, effective resistance, estimation, graph density, graph theory, scaling law, sensor network.**

I. Introduction

SEVERAL applications in sensor and actuator networks lead to estimation problems where a number of variables are to be estimated from noisy measurements of the difference between certain pairs of them. Consider the problem of localization, where a sensor does not know its position in a global coordinate system, but can measure its position relative to a set of nearby nodes. These measurements can be obtained, for example, from range and angle data but are typically subjected to large noise (see Fig. 1). In particular, two nearby sensors u and v located in a plane at positions x_u and x_v, respectively, have access to the measurement

$$\zeta_{u,v} = x_u - x_v + \epsilon_{u,v} \tag{1}$$

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where $\epsilon_{u,v}$ denotes measurement error. The problem of interest is to use the $\zeta_{u,v}$'s to estimate the positions of all the nodes in a common coordinate system whose origin is fixed arbitrarily at one of the nodes.

Similar estimation problems arise in time synchronization [1]–[3] and motion consensus in sensor-actuator networks [4]; see [4] and [5] for an overview of these applications. Motivated by these applications, we study the problem of estimating vector valued variables from noisy measurements of the difference between them. In particular, denoting the variables of interest by $\{x_i : i \in \mathcal{V}\}$ where $\mathcal{V} := \{1, 2, \ldots\}$, we consider problems for which noisy relative measurements of the form (1) are available. The ordered pairs of indices (u, v) for which we have relative measurements form a set $\mathcal{E}$ that is a (typically strict) subset of the set $\mathcal{V} \times \mathcal{V}$ of all pairs of indices. Just with relative measurements, the x_u's can be determined only up to an additive constant. To avoid this ambiguity, we assume that a particular variable (say x_o) is used as the *reference*, which is therefore assumed known. The problem of interest is to estimate the remaining node variables from all the available measurements.

The measurement (1) can be naturally associated with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with an edge from node u to v if the measurement $\zeta_{u,v}$ is available. The graph $\mathcal{G}$ is called the *measurement graph*, and each vector x_u, $u \in \mathcal{V}$, is called the *uth node variable*. The measurement noise $\epsilon_e, e \in \mathcal{E}$ is assumed zero mean and spatially uncorrelated, i.e., $\mathrm{E}[\epsilon_e] = 0 \forall e \in \mathcal{E}$ and $\mathrm{E}[\epsilon_e \epsilon_{\bar{e}}^T] = 0$ if $e \neq \bar{e}$.

In this paper, we investigate how the structure of the graph $\mathcal{G}$ affects the quality of the optimal linear unbiased estimate $\hat{x}_u$ of x_u, measured in terms of the covariance of the estimation error $\Sigma_{u,o} := \mathrm{E}[(x_u - \hat{x}_u)(x_u - \hat{x}_u)^T]$. The *optimal linear unbiased estimate* refers to the one obtained with the classical best linear unbiased estimator (BLUE), which achieves the minimum variance among all linear unbiased estimators [7]. We examine the growth of the BLUE error variance of a node u as a function of its distance to the reference node.

We are interested in the growth of error with distance in large graphs, for which infinite graphs (with a countably infinite number of nodes and edges) serve as proxies. This paper is focused on infinite graphs because the absence of boundary conditions in infinite graphs allows for more complete and simpler results. Using infinite graphs as proxies for large finite graphs is theoretically justified by the fact that the BLUE error variance of a node variable x_u in a large but finite subgraph of an infinite graph is arbitrarily close to the BLUE estimation error in the infinite graph, as long as the finite graph is sufficiently large. This convergence result was established in [8].

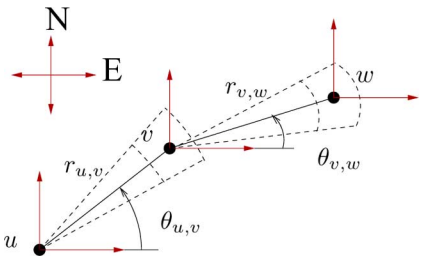

Fig. 1. Relative position measurement in a Cartesian reference frame using range and angle measurements. A local compass at each sensor is needed to measure bearing with respect to a common North. Noisy measurements of the range $r_{u,v}$ and angle $\theta_{u,v}$ between a pair of sensors u and v, which are denoted by $\hat{r}_{u,v}$ and $\hat{\theta}_{u,v}$, are converted to noisy measurements of relative position in the $x-y$ plane as $\zeta_{u,v} = \frac{1}{\bar{c}}[\hat{r}_{u,v}\cos\hat{\theta}_{u,v}, \hat{r}_{u,v}\sin\hat{\theta}_{u,v}]^T$, with $\bar{c} = \mathrm{E}[\cos(\delta\theta)]$, where $\delta_\theta = \theta - \hat{\theta}$ is the random error in the angle measurement (see [6] for more details). The same procedure is performed for every pair of sensors that can measure their relative range and angle. The task then is to estimate the positions of all the nodes with respect to an arbitrary node in the network from the relative position measurements.

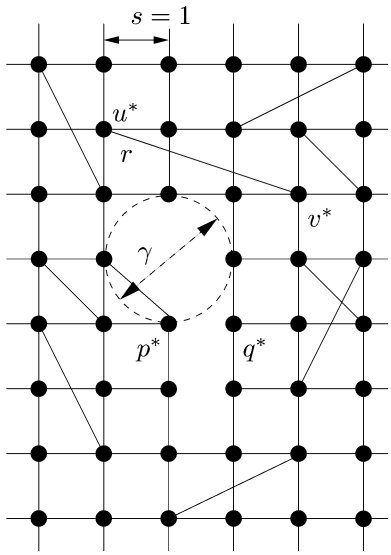

Fig. 2. Drawing of a graph in 2-D Euclidean space, and the corresponding denseness and sparseness parameters. Since the minimal distance between any two nodes is 1, the minimum node distance is $s = 1$. Since the longest edge is between u^* and v^*, the maximum connected range is $r = \sqrt{10}$. The diameter of the largest ball that can fit inside the drawing without enclosing any node is 2, so the maximum uncovered diameter is thus $\gamma = 2$. The minimal ratio between the Euclidean and graphical distance of a pair of nodes is achieved by the pair p^*, q^*, hence the asymptotic distance ratio is $\rho = d_f(p^*, q^*)/d_{\mathcal{G}}(p^*, q^*) = 1/5$.

When the measurement graph is a tree, there is a single path between the uth node and the reference node and one can show that the covariance matrix of the estimation error is the sum of the covariance matrices associated with this path. Thus, for trees, the variance of the BLUE estimation error of x_u grows linearly with the distance from node u to the reference node. It turns out that for graphs "denser" than trees, with multiple paths between pairs of nodes, the variance of the optimal linear unbiased estimation error can grow slower than linearly with distance.

In this paper, we introduce a novel notion of denseness for graphs that is needed to characterize how the estimation error grows with distance. In classical graph-theoretic terminology, a graph with n vertices is called dense if its average node degree is of order n, and is called sparse if its average node degree is a constant independent of n [9]. We recall that the degree of a node is the number of edges incident on it (an edge (u, v) is said to be incident on the nodes u and v). Other notions of denseness include geodenseness introduced by [10], which requires uniform node density (nodes per unit area) but does not consider the role of edges. Accuracy of localization from distance-only measurements have been extensively studied in the sensor networks literature, typically by evaluating the Cramér–Rao lower bound (see [5] and references therein). In many of these studies, graph density (as measured by node degree or node density) is recognized to affect estimation accuracy [12], [13], [15]. However, we will see through examples in Remark 2 that for the estimation problem considered in this paper, such notions of denseness are not sufficient to characterize how the estimation error grows with distance.

A key contribution of this paper is the development of suitable notions of graph denseness and sparseness that are useful in determining BLUE error scaling laws. These notions exploit the relationship between the measurement graph and a lattice. We recall that the d-dimensional square lattice $\mathbf{Z}_d$ is defined as a graph with a node in every point in $\mathbb{R}^d$ with integer coordinates and an edge between every pair of nodes at an Euclidean distance of 1 (see Fig. 4 for examples). The error scaling laws for a lattice measurement graph can be determined analytically by exploiting symmetry. It turns out that when the graph is not a lattice, it can still be compared to a lattice. Intuitively, if after some bounded perturbation in its node and edge set, the graph looks approximately like a d-dimensional lattice, then the graph inherits the denseness properties of the lattice. In that case, the error covariance for the lattice can still be used to bound the error covariance in the original graph.

Our classification of dense and sparse graphs in $\mathbb{R}^d, d \in \{1,2,3\}$, characterizes BLUE error scaling laws. For dense graphs, they provide upper bounds on the growth rate of the error, while for sparse graphs, they provide lower bounds. The precise growth rates depend on which dimension the graph is dense or sparse in. When a graph is dense in 1-D, 2-D, or 3-D, respectively, the error covariance of a node is upper bounded by a linear, logarithmic, or bounded function, respectively, of its distance from the reference. On the other hand, when a graph is sparse in 1-D, 2-D, or 3-D, the error covariance of a node is lower bounded by a linear, logarithmic, and bounded function, respectively, of its distance from the reference. Our sparse graphs are also known as "graphs that can be drawn in a civilized manner" according to the terminology introduced by [16] in connection with random walks.

The BLUE error scaling laws derived in this paper provide an algorithm-independent limit to the estimation accuracy achievable in large networks, since no linear unbiased estimation algorithm can achieve higher accuracy than the BLUE estimator. For example, when a graph is sparse in 1-D, the BLUE estimation error covariance grows at least linearly with the distance from the reference. Therefore, the estimation accuracy will be necessarily poor in large 1-D sparse graphs. On the other hand, when a graph is dense in 3-D, the BLUE estimation error of every

node variable remains below a constant, even for nodes that are arbitrarily far away from the reference. So accurate estimation is possible in very large 3-D dense graphs.

The results in this paper are useful for the design and deployment of *ad hoc* and sensor networks. Since we now know what structural properties are beneficial for accurate estimation, we can strive to achieve those structures when deploying a network. Specifically, we should try to achieve a dense-in-$\mathbb{R}^d$ structure, with d as large as possible, for high accuracy estimation. Since the scaling laws are true for the optimal linear unbiased estimator, they can also help designers determine if design requirements are achievable. For example, if the requirement is that the estimation accuracy should not decrease with size, no matter how large a network is, the network must be dense in $\mathbb{R}^d, d \geq 3$ for such a requirement to be satisfied.

Our results also expose certain misconceptions that exist in the sensor network literature about the relationship between graph structure and estimation error. In Section II-B, we provide examples that expose the inadequacy of the usual measures of graph denseness, such as node degree, in determining scaling laws of the estimation error.

In practice, more than one reference node (commonly referred to as anchors) may be used. We only consider the case of a single reference node since scaling laws with a single reference provide information on how many reference nodes may be needed. For example, since the estimation error in a 3-D dense graph is bounded by a constant, one reference node may be enough for such a graph.

While we do not discuss the computation of the optimal linear unbiased estimates in this paper, we have developed distributed algorithms to compute these estimates with arbitrary precision (see [11]–[15] and references therein). These algorithms are distributed in the sense that every node computes its own estimate and the information needed to carry out this computation is obtained by communication with its neighbors.

A preliminary version of some of the results in this paper was presented in [17]. However, stricter assumptions to establish the upper bounds on error growth rates were used in [17]. Moreover, only sufficient conditions were obtained in [17] for some of the error scaling laws to hold; whereas here we derive necessary and sufficient conditions.

Organization

The rest of this paper is organized as follows. Section II describes the problem and summarizes the main results of the paper. Section III describes key properties of dense and sparse graphs. Section IV briefly describes the analogy between BLUE and generalized electrical networks from [8] that is needed to prove the main results. Section V contains the proof of the main result of the paper. Section VI deals with the question of how to check if a graph possesses the denseness/sparseness properties. The paper ends with a few final conclusions and directions for future research in Section VII.

II. Problem Statement and Main Results

Recall that we are interested in estimating vector-valued variables $x_u \in \mathbb{R}^k, u \in \mathcal{V} := \{1, 2, \ldots\}$, from noisy relative measurements of the form

$$\zeta_{u,v} = x_u - x_v + \epsilon_{u,v}, \qquad (u,v) \in \mathcal{E} \tag{2}$$

where $\epsilon_{u,v}$ denotes a zero-mean measurement noise and $\mathcal{E}$ is the set of ordered pairs (u,v) for which relative measurements are available. The node set $\mathcal{V}$ is either finite, or infinite, but countable. We assume that the value of a particular *reference variable* x_o is known and without loss of generality we take $x_o = 0$. The node set $\mathcal{V}$ and the edge set $\mathcal{E}$ together define a directed measurement graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$.

The accuracy of a node variable's estimate, measured in terms of the covariance of the estimation error, depends on the graph $\mathcal{G}$ as well as the measurement errors. The covariance matrix of the error $\epsilon_{u,v}$ in the measurement $\zeta_{u,v}$ is denoted by $P_{u,v}$, i.e., $P_{u,v} := \mathrm{E}[\epsilon_{u,v}\epsilon_{u,v}^T]$. We assume that the measurement errors on different edges are uncorrelated, i.e., for every pair of distinct edges $e, \bar{e} \in \mathcal{E}, \mathrm{E}[\epsilon_e \epsilon_{\bar{e}}^T] = 0$. The estimation problem is now formulated in terms of a *network* $(\mathcal{G}, P)$ where $P : \mathcal{E} \to \mathbb{S}^{k+}$ is a function that assigns to each edge $(u,v) \in \mathcal{E}$ the covariance matrix $P_{u,v}$ of the measurement error associated with the edge (u,v) in the measurement graph $\mathcal{G}$. The symbol $\mathbb{S}^{k+}$ denotes the set of $k \times k$ symmetric positive–definite matrices.

As discussed in Section I, our results are stated for infinite networks. The following conditions are needed to make sure that the estimation problem is well posed and that the estimates satisfy appropriate convergence properties to be discussed shortly.

Assumption 1 (Measurement Network): The measurement network $(\mathcal{G}, P)$ satisfies the following properties.

1) The graph $\mathcal{G}$ is weakly connected, i.e., it is possible to go from every node to every other node traversing the graph edges without regard to edge direction.
2) The graph $\mathcal{G}$ has a finite maximum node degree.[1]
3) The edge-covariance function P is uniformly bounded, i.e., there exist constant symmetric positive definite matrices $P_{\min}, P_{\max}$ such that $P_{\min} \leq P_e \leq P_{\max}, \forall e \in \mathcal{E}$. □

In the above, for two matrices $A, B \in \mathbb{R}^{k \times k}, A > B (A \geq B)$ means $A - B$ is positive definite (semidefinite). We write $A < B (A \leq B)$ if $-A > -B (-A \geq -B)$.

We also assume throughout this paper that measurement graphs do not have parallel edges. A number of edges are said to be parallel if all of them are incident on the same pair of nodes. The condition of not having parallel edges is not restrictive since parallel measurements can be combined into a single measurement with an appropriate covariance, while preserving the BLUE error covariances (see Remark 3).

Given a finite measurement network $(\mathcal{G}_{\text{finite}}, P)$, where $\mathcal{G}_{\text{finite}}$ contains the nodes u and o, it is straightforward to compute the BLUE estimate $\hat{x}_u(\mathcal{G}_{\text{finite}})$ of the unknown node variable x_u in the network $(\mathcal{G}_{\text{finite}}, P)$, as described in [8], and the covariance matrix of the estimation error $\Sigma_{u,o}(\mathcal{G}_{\text{finite}}) := \mathrm{E}[(x_u - \hat{x}_u)(x_u - \hat{x}_u)^T]$ exists as long as $\mathcal{G}_{\text{finite}}$ is weakly connected [8]. Due to the optimality of the

[1]The degree of a node is the number of edges that is incident on the node. An edge (u,v) is said to be incident on the nodes u and v.

BLUE, $\Sigma_{u,o}(\mathcal{G}_{\text{finite}})$ is the minimum possible estimation error covariance that is achievable by any linear unbiased estimator using all the measurements in the graph $\mathcal{G}_{\text{finite}}$.

When the measurement graph is infinite, the BLUE error covariance $\Sigma_{u,o}$ for a node variable x_u is defined as

$$\Sigma_{u,o} = \inf_{\mathcal{G}_{\text{finite}}} \Sigma_{u,o}(\mathcal{G}_{\text{finite}}) \tag{3}$$

where the infimum is taken over all finite subgraphs $\mathcal{G}_{\text{finite}}$ of $\mathcal{G}$ that contain the nodes u and o. We define a matrix M to be the *infimum* of the matrix set $S \subset \mathbb{S}^{k+}$, and denote it by

$$M = \inf S \tag{4}$$

if $M \leq A$ for every matrix $A \in S$, and for every positive real ϵ, there exists a matrix $B \in S$ such that $M + \epsilon I_k > B$. Under Assumption 1, it was shown in [8] that the infimum in (3) always exists. In this case, (3) means that the BLUE covariance $\Sigma_{u,o}$ is the lowest error covariance that can be achieved by using all the available measurements.

In the sequel, we determine how the BLUE covariance $\Sigma_{u,o}$ grows as a function of the distance of node u to the reference o, and how this scaling law depends on the structure of the measurement graph $\mathcal{G}$. To this effect we start by providing a classification of graphs that is needed to characterize the error scaling laws.

A. Graph Denseness and Sparseness

We start by introducing graph drawings, which will later allow us to define dense and sparse graphs.

1) Graph Drawings: The drawing of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in a d-dimensional Euclidean space is obtained by mapping the nodes into points in $\mathbb{R}^d$ by a *drawing function* $f : \mathcal{V} \to \mathbb{R}^d$. A drawing is also called a representation [18] or an embedding [9]. For a particular drawing f, given two nodes $u, v \in \mathcal{V}$ the *Euclidean distance between* u and v *induced by the drawing* $f : \mathcal{V} \to \mathbb{R}^d$ is defined by

$$d_f(u,v) := ||f(v) - f(u)||$$

where $||\cdot||$ denoted the usual Euclidean norm in $\mathbb{R}^d$. It is important to emphasize that the definition of drawing allows edges to intersect and therefore every graph has a drawing in every Euclidean space. In fact, every graph has an infinite number of drawings in every Euclidean space. However, a particular drawing is useful only if it clarifies the relationship between the graph and the Euclidean space in which it is drawn. In what follows, given two nodes u and v, $d_{\mathcal{G}}(u,v)$ denotes the *graphical distance* between u and v, i.e., the number of edges in the shortest path between u and v. The graphical distance $d_{\mathcal{G}}$ is evaluated without regards to edge directions, which are immaterial in determining BLUE error covariances (see Remark 4).

For a particular drawing f and induced Euclidean distance d_f of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, four parameters are needed to characterize graph denseness and sparseness. The *minimum node distance*, denoted by s, is defined as the minimum Euclidean distance between the drawing of two nodes

$$s := \inf_{\substack{u,v \in \mathcal{V} \\ v \neq u}} d_f(u,v).$$

The *maximum connected range*, denoted by r, is defined as the Euclidean length of the drawing of the longest edge

$$r := \sup_{(u,v) \in \mathcal{E}} d_f(u,v).$$

The *maximum uncovered diameter*, denoted by γ, is defined as the diameter of the largest open ball that can be placed in $\mathbb{R}^d$ such that it does not enclose the drawing of any node

$$\gamma := \sup \{\delta : \exists \mathcal{B}_\delta \text{ s.t.} f(u) \notin \mathcal{B}_\delta, \forall u \in \mathcal{V}\}$$

where the existential quantification spans over the balls $\mathcal{B}_\delta$ in $\mathbb{R}^d$ with diameter δ and centered at arbitrary points. Finally, the *asymptotic distance ratio*, denoted by ρ, is defined as

$$\rho := \lim_{n \to \infty} \inf \left\{ \frac{d_f(u,v)}{d_{\mathcal{G}}(u,v)} : u, v \in \mathcal{V} \text{ and } d_{\mathcal{G}}(u,v) \geq n \right\}.$$

Essentially ρ provides a lower bound for the ratio between the Euclidean and the graphical distance for nodes that are far apart. The asymptotic distance ratio can be thought of as an inverse of the *stretch* for geometric graphs, which is a well-studied concept for finite graphs [19].

The two parameters ρ and r defined above are especially useful to compare graphical and Euclidean distances, as stated in the following result.

Lemma 1 (Euclidean Versus Graphical Distances): The following two statements are equivalent.

1) The asymptotic distance ratio ρ is strictly positive.
2) There exist constants $\alpha > 0, \beta > 0$ for which

$$d_{\mathcal{G}}(u,v) \leq \alpha d_f(u,v) + \beta \qquad \forall u, v \in \mathcal{V}. \tag{5}$$

Similarly, the following statements are equivalent.

1) The maximum connected range r is finite.
2) There exist constants $\alpha > 0, \beta \geq 0$ for which

$$d_f(u,v) \leq \alpha d_{\mathcal{G}}(u,v) + \beta \qquad \forall u, v \in \mathcal{V}. \qquad \square$$

The proof of this lemma is provided in the Appendix.

2) Dense and Sparse Graphs: We call the drawing of a graph with finite maximum uncovered diameter ($\gamma < \infty$) and positive asymptotic distance ratio ($\rho > 0$) a *dense drawing*. We say that a graph $\mathcal{G}$ is dense in $\mathbb{R}^d$ if there exists a dense drawing of the graph in $\mathbb{R}^d$. Graph drawings for which the minimum node distance is positive ($s > 0$) and the maximum connected range is finite ($r < \infty$) are called *civilized drawings* [16]. A graph $\mathcal{G}$ is said to be *sparse in* $\mathbb{R}^d$ if there exists a civilized drawing in $\mathbb{R}^d$.

It follows from these definitions and Lemma 1 that if a graph is dense in $\mathbb{R}^d$, then it has enough nodes and edges so that it is possible to draw it in $\mathbb{R}^d$ in such a way that its nodes cover $\mathbb{R}^d$ without leaving large holes (finite γ), and yet a small Euclidean distance between two nodes in the drawing guarantees a small graphical distance between them [positive ρ, which implies (5)]. On the other hand, if a graph is sparse in $\mathbb{R}^d$, then one can draw it in $\mathbb{R}^d$ so as to keep a certain minimum separation between nodes (positive s) without making the edges arbitrarily long (finite r). It also follows from the definitions that a graph must be infinite

to be dense in any dimension, and a finite graph is sparse in every dimension.

A graph can be both dense and sparse in the same dimension. For example, the d-dimensional lattice is both sparse and dense in $\mathbb{R}^d$. However, there is no civilized drawing of the d-dimensional lattice in $\mathbb{R}^{d'}$ for any $d' < d$. Moreover, there is no dense drawing of the d-dimensional lattice in $\mathbb{R}^{\bar{d}}$ for every $\bar{d} > d$. This means, for example, that the 3-D lattice in not sparse in 2-D and is not dense in 4-D. In general, a graph being dense in a particular dimension puts a restriction on which dimensions it can be sparse in. The next result, proved in Section VI, states this precisely.

Lemma 2: A graph that is dense in $\mathbb{R}^d$ for some $d \geq 2$ cannot be sparse in $\mathbb{R}^{d'}$ for every $d' < d$. □

Remark 1 (Historical Note): In the terminology of [16], sparse graphs (as defined here) are said to be graphs "that can be drawn in a civilized manner." In this paper, we refer to such graphs as sparse graphs since they are the antitheses of dense graphs. □

B. Error Scaling Laws

The concepts of dense and sparse graphs allow one to characterize precisely how the BLUE error covariance $\Sigma_{u,o}$ grows with the distance from the node u to the reference o. The next theorem, which establishes the BLUE error scaling laws for dense and sparse graphs, is the main result of the paper. The proof of the theorem is provided in Section V.

Before we present the theorem, we need to introduce some notation. The asymptotic notations $\Omega(\cdot)$ and $\mathsf{O}(\cdot)$ are used for matrix valued functions in the following way. For a matrix-valued function $g : \mathbb{R} \to \mathbb{R}^{k\times k}$ and a scalar-valued function $p : \mathbb{R} \to \mathbb{R}$, the notation $g(x) = \mathcal{O}(p(x))$ means that there exist a positive constant x_o and a constant matrix $A \in \mathbb{S}^{k+}$ such that $g(x) \leq Ap(x)$ for all $x > x_o$. Similarly, $g(x) = \Omega(p(x))$ means there exist a positive constant x_o and a constant matrix $B \in \mathbb{S}^{k+}$ such that $g(x) \geq Bp(x)$ for all $x > x_o$. Recall that $\mathbb{S}^{k+}$ is the set of all $k \times k$ symmetric positive–definite matrices.

Theorem 1 (Error Scaling Laws): Consider a measurement graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that satisfies Assumption 1, with a reference node $o \in \mathcal{V}$. The BLUE error covariance $\Sigma_{u,o}$ for a node u obeys the scaling laws shown in Table I. □

A graph can be both sparse and dense in a particular dimension, in which case the asymptotic upper and lower bounds are the same. For a graph that is both sparse and dense in $\mathbb{R}^d$, the error covariance grows with distance in the same rate as it does in the corresponding lattice $\mathbf{Z}_d$.

Remark 2 (Counterexamples to Conventional Wisdom): As noted in Section I, the average node degree of a graph or the number of nodes and edges per unit area of a deployed network are often used as measures of graph denseness. However, these measures do not predict error scaling laws. The three graphs in Fig. 3 offer an example of the inadequacy of node degree as a measure of denseness. This figure shows a 3-fuzz of the 1-D lattice (see Section III for the formal definition of a fuzz), a triangular lattice, and a 3-D lattice. It can be verified from the definitions in Section II-A2 that the 3-fuzz of the 1-D lattice is both dense and sparse in $\mathbb{R}$, the triangular lattice is dense and sparse in $\mathbb{R}^2$, and the 3-D lattice is dense and sparse in $\mathbb{R}^3$. Thus, it follows from Theorem 1 that the BLU estimation error scales linearly with distance in the 3-fuzz of the 1-D lattice, logarithmically with distance in the triangular lattice, and is uniformly bounded with respect to distance in the 3-D lattice, even though every node in each of these graphs has the same degree, namely six. □

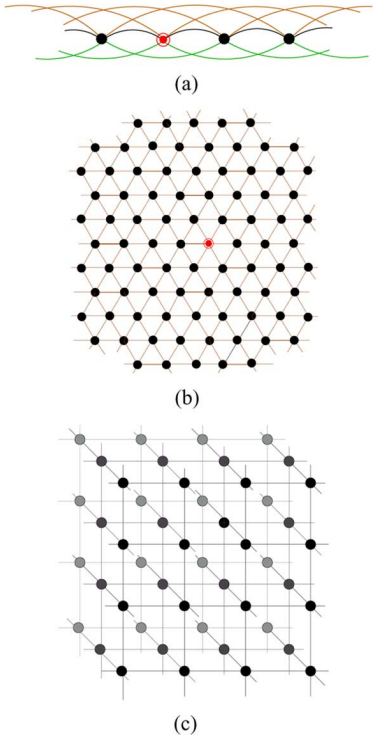

Fig. 3. Three measurement graphs that show vastly different scaling laws of the estimation error, whereas each has the same node degree for every node. Furthermore, they are all "sparse" according to traditional graph-theoretic terminology (see the discussion on graph denseness in Section I).

We note that the notion of geodenseness introduced in [10] is also not useful for characterizing error scaling laws since geodenseness considers node density alone without regard to the edges.

III. Dense and Sparse Graphs

This section establishes an embedding relationship between dense and sparse graphs and lattices, which is needed to prove Theorem 1. Roughly speaking, a graph $\mathcal{G}$ can be embedded in another graph $\bar{\mathcal{G}}$ if $\bar{\mathcal{G}}$ contains all the nodes and edges of $\mathcal{G}$, and perhaps a few more. The usefulness of embedding in answering the error scaling question is that when $\mathcal{G}$ can be embedded in $\bar{\mathcal{G}}$, the BLUE error covariances in $\mathcal{G}$ are larger than the corresponding ones in $\bar{\mathcal{G}}$ (this statement will be made precise in Theorem 5 of Section IV).

The *h-fuzz of a graph* $\mathcal{G}$, introduced by Doyle and Snell [16], is a graph with the same set of nodes as $\mathcal{G}$ but with a larger set of edges. Specifically, given a graph $\mathcal{G}$ and a positive integer h,

TABLE I
COVARIANCE $\Sigma_{u,o}$ OF x_u'S BLUE ESTIMATION ERROR FOR GRAPHS THAT ARE DENSE OR SPARSE IN $\mathbb{R}^d$. IN THE TABLE, $d_{f_d}(u,o)$ DENOTES THE EUCLIDEAN DISTANCE BETWEEN NODE u AND THE REFERENCE NODE o INDUCED BY A DRAWING $f_d \cdot \mathcal{V} \to \mathbb{R}^d$ THAT ESTABLISHES THE GRAPH'S DENSENESS IN THE EUCLIDEAN SPACE $\mathbb{R}^d$, AND $d_{f'_d}(u,o)$ DENOTES THE EUCLIDEAN DISTANCE INDUCED BY A DRAWING $f'_d \cdot \mathcal{V} \to \mathbb{R}^d$ THAT ESTABLISHES THE GRAPH'S SPARSENESS

Euclidean space and graph example	Covariance matrix $\Sigma_{u,o}$ of the estimation error of x_u in a *sparse graph* with a sparse drawing f'_d	Covariance matrix $\Sigma_{u,o}$ of the estimation error of x_u in a *dense graph* with a dense drawing f_d
 $\mathbb{R}$	$\Sigma_{u,o}(\mathcal{G}) = \Omega\Big(d_{f'_1}(u,o)\Big)$	$\Sigma_{u,o}(\mathcal{G}) = \mathcal{O}\Big(d_{f_1}(u,o)\Big)$
$\mathbb{R}^2$	$\Sigma_{u,o}(\mathcal{G}) = \Omega\Big(\log d_{f'_2}(u,o)\Big)$	$\Sigma_{u,o}(\mathcal{G}) = \mathcal{O}\Big(\log d_{f_2}(u,o)\Big)$
 $\mathbb{R}^3$	$\Sigma_{u,o}(\mathcal{G}) = \Omega\Big(1\Big)$	$\Sigma_{u,o}(\mathcal{G}) = \mathcal{O}\Big(1\Big)$

an h-fuzz of $\mathcal{G}$, denoted by $\mathcal{G}^{(h)}$, is a graph that has an edge between two nodes u and v whenever the graphical distance between these nodes in $\mathcal{G}$ is less than or equal to h.

We say that a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be *embedded* in another graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ if $\mathcal{V} \subset \bar{\mathcal{V}}$, and, whenever there is an edge between two nodes in $\mathcal{G}$, there is an edge between them in $\bar{\mathcal{G}}$. More precisely, $\mathcal{G}$ can be embedded in $\bar{\mathcal{G}}$ if there exists an injective map $\eta : \mathcal{V} \to \bar{\mathcal{V}}$ such that for every $(u,v) \in \mathcal{E}$, either $(\eta(u), \eta(v)) \in \bar{\mathcal{E}}$ or $(\eta(v), \eta(u)) \in \bar{\mathcal{E}}$. In the sequel, we use $\mathcal{G} \subset \bar{\mathcal{G}}$ to denote that $\mathcal{G}$ can be embedded in $\bar{\mathcal{G}}$.

A. Relationship With Lattices and Euclidean Spaces

The next theorems (Theorem 2 and 3) show that sparse graphs can be embedded in fuzzes of lattices, and fuzzes of dense graphs can embed lattices. In these two theorems, we use $d_{\mathbf{Z}_d}(\cdot)$ to denote the graphical distance in the lattice $\mathbf{Z}_d$ and $d_f(\cdot)$ to denote the Euclidean distance in $\mathbb{R}^d$ induced by the drawing f.

Theorem 2 (Sparse Embedding): A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is sparse in $\mathbb{R}^d$ if and only if there exists a positive integer h such that $\mathcal{G} \subset \mathbf{Z}_d^{(h)}$. Moreover, if $f : \mathcal{V} \to \mathbb{R}^d$ is a civilized drawing of $\mathcal{G}$ in $\mathbb{R}^d$, then there exists an embedding $\eta : \mathcal{V} \to \mathcal{V}_{\mathbf{Z}_d}$ so that $\forall u, v \in \mathcal{V}$

$$d_{\mathbf{Z}_d}(\eta(u), \eta(v)) \geq \sqrt{d}\left(\frac{1}{s} d_f(u,v) - 2\right) \tag{6}$$

where s is the minimum node distance in the f-drawing of $\mathcal{G}$. □

In words, the theorem states that $\mathcal{G}$ is sparse in $\mathbb{R}^d$ if and only if $\mathcal{G}$ can be embedded in an h-fuzz of a d-dimensional lattice. The significance of the additional condition (6) is that if the Euclidean distance between a pair of nodes u and v in a civilized drawing of the graph is large, the graphical distance in the lattice between the nodes that correspond to u and v must also be large.

The first statement of Theorem 2 is essentially taken from [16], where it was proved that if a graph can be drawn in a civilized manner in $\mathbb{R}^d$, then it can be embedded in an h-fuzz of a d-lattice, where h depends only on s and r. A careful examination of the proof in [16] reveals that it is not only sufficient but also a necessary condition for embedding in lattice fuzzes. The proof of this theorem is therefore omitted.

Theorem 3 (Dense Embedding): A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is dense in $\mathbb{R}^d$ if and only if there exists finite, positive integers h and c such that the following conditions are satisfied:

i) $\mathcal{G}^{(h)} \supset \mathbf{Z}_d$;

ii) if $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$ is an embedding of $\mathbf{Z}_d$ into $\mathcal{G}^{(h)}$, then $\forall u \in \mathcal{V}, \exists \bar{u} \in \eta(\mathcal{V}_{\mathbf{Z}_d}) \subseteq \mathcal{V}$ such that $d_{\mathcal{G}}(u, \bar{u}) \leq c$.

Moreover, if $f : \mathcal{V} \to \mathbb{R}^d$ is a dense drawing of $\mathcal{G}$ in $\mathbb{R}^d$, then the embedding function η in ii) can be chosen so that $\forall u, v \in \mathcal{V}$, we can find $u_z, v_z \in \mathcal{V}_{\mathbf{Z}_d}$ satisfying

$$\begin{aligned} &d_{\mathcal{G}}(u, \eta(u_z)) \leq c, \quad d_{\mathcal{G}}(v, \eta(v_z)) \leq c \\ &d_{\mathbf{Z}_d}(u_z, v_z) \leq 4d + \frac{\sqrt{d}}{\gamma} d_f(u,v) \end{aligned} \tag{7}$$

where γ is the maximum uncovered diameter of the f-drawing of $\mathcal{G}$. □

In other words, the two conditions state that $\mathcal{G}$ is dense in $\mathbb{R}^d$ if and only if i) the d-dimensional lattice can be embedded in an h-fuzz of $\mathcal{G}$ for some positive integer h and ii) every node of $\mathcal{G}$ that is not the image of a node in $\mathbf{Z}_d$ is at a uniformly bounded graphical distance from a node that is the image of a node in $\mathbf{Z}_d$. The significance of (7) is that not only we can find for every node in $\mathcal{G}$ a close-by node that has a preimage in the lattice, but also these close-by nodes can be so chosen so that if the Euclidean distance between a pair of nodes u and v in the drawing is small, then the graphical distance in the lattice between the preimages of their close-by nodes is small as well.

IV. ELECTRICAL ANALOGY

A crucial step in proving the main results of this paper is the analogy introduced in [8] between the BLUE problem and an abstract electrical network, where currents, potentials and resistances are matrix valued.

A *generalized electrical network* $(\mathcal{G}, R)$ consists of a graph $\mathcal{G} = (\mathbf{V}, \mathbf{E})$ (finite or infinite) together with a function $R : \mathcal{E} \to \mathbb{S}^{k+}$ that assigns to each edge $e \in \mathcal{E}$ a symmetric positive–definite matrix R_e called the *generalized resistance* of the edge.

A generalized flow from node $u \in \mathcal{V}$ to node $v \in \mathcal{V}$ with intensity $\mathbf{j} \in \mathbb{R}^{k\times k}$ is an edge-function $j : \mathcal{E} \to \mathbb{R}^{k\times k}$ such that

$$\sum_{\substack{(p,q)\in\mathbf{E}\\ p=\bar{p}}} j_{p,q} - \sum_{\substack{(q,p)\in\mathbf{E}\\ p=\bar{p}}} j_{q,p} = \begin{cases} \mathbf{j}, & \bar{p} = u \\ -\mathbf{j}, & \bar{p} = v \\ \mathbf{0}, & \text{otherwise} \end{cases} \qquad \forall \bar{p} \in \mathcal{V}. \tag{8}$$

A flow j is said to have finite support if it is zero on all but a finite number of edges. We say that a flow i is a *generalized current* when there exists a *node-function* $V : \mathbf{V} \to \mathbb{R}^{k\times k}$ for which

$$R_{u,v} i_{u,v} = V_u - V_v \qquad \forall (u,v) \in \mathcal{E}. \tag{9}$$

The node-function V is called a *generalized potential associated with the current* i. Equation (8) should be viewed as a generalized version of Kirchhoff's current law and can be interpreted as: the net flow out of each node other than u and v is equal to zero, whereas the net flow out of u is equal to the net flow into v and both are equal to the flow intensity $\mathbf{j}$. Equation (9) provides in a combined manner a generalized version of Kirchhoff's loop law, which states that the net potential drop along a circuit must be zero, and Ohm's law, which states that the potential drop across an edge must be equal to the product of its resistance and the current flowing through it. A circuit is an undirected path that starts and ends at the same node. For $k = 1$, generalized electrical networks are the usual electrical networks with scalar currents, potentials, and resistors.

A. Effective Resistance and BLUE Error Covariance

It was shown in [8] that when a current of intensity $\mathbf{i} \in \mathbb{R}^{k\times k}$ flows from node u to node v, the resulting generalized current i is a linear function of the intensity $\mathbf{i}$ and there exists a matrix $R^{\text{eff}}_{u,v} \in \mathbb{S}^{k+}$ such that

$$V_u - V_v = R^{\text{eff}}_{u,v}\mathbf{i} \qquad \forall \mathbf{i} \in \mathbb{R}^{k\times k}. \tag{10}$$

We call the matrix $R^{\text{eff}}_{u,v}$ the *generalized effective resistance between* u and v. In view of this definition, the effective resistance between two nodes is the generalized potential difference between them when a current with intensity equal to the identity matrix I_k is injected at one node and extracted at the other, which is analogous to the definition of effective resistance in scalar networks [16]. Note that the effective resistance between two arbitrary nodes in a generalized network is a symmetric positive–definite matrix as long as the network satisfies Assumption 1, whether the network is finite or infinite [8].

Generalized electrical networks are useful in studying the BLUE error in large networks because of the following analogy between the BLUE error covariance and the generalized effective resistance.

Theorem 4 (Electrical Analogy [8]): Consider a measurement network $(\mathcal{G}, P)$ satisfying Assumption 1 with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a single reference node $o \in \mathcal{V}$. Then, for every node $u \in \mathcal{V} \setminus \{o\}$, the BLUE error covariance $\Sigma_{u,o}$ defined in (3) is a symmetric positive–definite matrix equal to the generalized effective resistance $R^{\text{eff}}_{u,o}$ between u and o in the generalized electrical network $(\mathcal{G}, P)$

$$\Sigma_{u,o} = R^{\text{eff}}_{u,o}. \qquad \square$$

Remark 3: In an electrical network, parallel resistors can be combined into one resistor by using the parallel resistance formula so that the effective resistance between every pair of nodes in the network remain unchanged. The same can be done in generalized electrical networks [20]. The analogy between BLUE covariance and effective resistance means that parallel measurement edges with possibly distinct measurement error covariances can be replaced by a single edge with an equivalent error covariance, so that the BLUE error covariances of all nodes remain unchanged. This explains why the assumption of not having parallel edges made at the beginning is not restrictive in any way.

B. Graph Embedding and Partial Ordering of BLUE Covariances

Effective resistance in scalar electrical networks satisfies Rayleigh's monotonicity law, which states that the effective resistance between any two nodes can only increase if the resistance on any edge is increased, and *vice versa* [16]. The next result (proved in [8]) states that the same is true for generalized networks, whether finite or infinite.

Theorem 5 (Rayleigh's Monotonicity Law [8]): Consider two generalized electrical networks $(\mathcal{G}, R)$ and $(\bar{\mathcal{G}}, \bar{R})$ with graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, respectively, such that both networks satisfy Assumption 1. Assume that:

1) $\mathcal{G}$ can be embedded in $\bar{\mathcal{G}}$, i.e., $\mathcal{G} \subset \bar{\mathcal{G}}$;
2) $R_e \geq \bar{R}_{\bar{e}}$ for every edge $e \in \mathcal{E}$.

Then, for every pair of nodes $u, v \in \mathcal{V}$ of $\mathcal{G}$

$$R^{\text{eff}}_{u,v} \geq \bar{R}^{\text{eff}}_{u,v}$$

where $R^{\text{eff}}_{u,v}$ and $\bar{R}^{\text{eff}}_{u,v}$ are the effective resistance between u and v in the networks $(\mathcal{G}, R)$ and $(\bar{\mathcal{G}}, \bar{R})$, respectively. $\square$

The usefulness of Rayleigh's monotonicity law in answering the error scaling question becomes apparent when combined with the electrical analogy. It shows that when $\mathcal{G}$ can be embedded in $\bar{\mathcal{G}}$, the BLUE error covariances in $\mathcal{G}$ are lower bounded by the error covariances in $\bar{\mathcal{G}}$. Intuitively, since $\mathcal{G}$ has only a subset of the measurements in $\bar{\mathcal{G}}$, the estimates in $\mathcal{G}$ are less accurate than those in $\bar{\mathcal{G}}$.

Remark 4: Although the graph $\mathcal{G}$ that defines the electrical network $(\mathcal{G}, R)$ is directed, the edge directions are irrelevant in determining effective resistances. This is why Rayleigh's monotonicity law holds with graph embedding, which is insensitive to edge directions. The electrical analogy also explains why the edge directions are irrelevant in determining error covariances. $\square$

C. Triangle Inequality

Matrix-valued effective resistances satisfy a triangle inequality, which will be useful in proving the error scaling laws in Section V. It is known that scalar effective resistance obeys triangle inequality, and is therefore also referred to as the "resistance distance" [21]. Although the result in [21] was proved only for finite networks, it is not hard to extend it to infinite networks. The following simple extension of the triangle inequality to generalized networks with constant resistances on every edge was derived in [20].

Lemma 3 (Triangle Inequality): Let $(\mathcal{G}, R_o)$ be a generalized electrical network satisfying Assumption 1 with a constant resistance $R_o \in \mathbb{S}^{k+}$ on every edge of $\mathcal{G}$. Then, for every triple of nodes u, v, w in the network

$$R^{\text{eff}}_{u,w} \leq R^{\text{eff}}_{u,v} + R^{\text{eff}}_{v,w}. \qquad \square$$

D. Effective Resistances in Lattices and Fuzzes

Recall that given a graph $\mathcal{G}$ and a positive integer h, the h-fuzz of $\mathcal{G}$, denoted by $\mathcal{G}^{(h)}$, is a graph that has an edge between two nodes u and v whenever the graphical distance between them in $\mathcal{G}$ is less than or equal to h.

An h-fuzz will clearly have lower effective resistance than the original graph because of Rayleigh's monotonicity law, but it is lower only by a constant factor as stated in the following result, which is a straightforward extension to the generalized case of a result about scalar effective resistance established by Doyle and Snell (see the Theorem on p. 103 as well as Exercise 2.4.9 in [16]). The interested reader can find a proof in [20].

Lemma 4: Let $(\mathcal{G}, R_o)$ be a generalized electrical network satisfying Assumption 1 with a constant generalized resistance $R_o \in \mathbb{S}^{k+}$ on its every edge. Let $(\mathcal{G}^{(h)}, R_o)$ be the electrical network similarly constructed on $G^{(h)}$, the h-fuzz of G. For every pair of nodes u and v in $\mathcal{V}$

$$\alpha R^{\text{eff}}_{u,v}(\mathcal{G}) \leq R^{\text{eff}}_{u,v}(\mathcal{G}^{(h)}) \leq R^{\text{eff}}_{u,v}(\mathcal{G})$$

where $R^{\text{eff}}_{u,v}(\,\cdot\,)$ is the effective resistance in the network $(\cdot, R_o)$ and $\alpha \in (0, 1]$ is a positive constant that does not depend on u and v. $\square$

The following lemma establishes effective resistances in d-dimensional lattices and their fuzzes.

Lemma 5: For a given positive integer h, consider the electrical network $(\mathbf{Z}^{(h)}_d, R_o)$ with a constant generalized resistance $R_o \in \mathbb{S}^{k+}$ at every edge of the h-fuzz of the d-dimensional square lattice $\mathbf{Z}_d$. The generalized effective resistance $R^{\text{eff}}_{u,v}$ between two nodes u and v in the electrical network $(\mathbf{Z}^{(h)}_d, R_o)$ satisfies:

1) $R^{\text{eff}}_{u,v}(\mathbf{Z}^{(h)}_1) = \Theta(d_{\mathbf{Z}_1}(u, v))$;
2) $R^{\text{eff}}_{u,v}(\mathbf{Z}^{(h)}_2) = \Theta(\log d_{\mathbf{Z}_2}(u, v))$;
3) $R^{\text{eff}}_{u,v}(\mathbf{Z}^{(h)}_3) = \Theta(1)$. $\square$

Proof of Lemma 5: The *scalar* effective resistance in 1-D, 2-D, and 3-D lattices follow linear, logarithmic, and bounded growth rates, respectively [22], [23]. Using these results, it was established in [8] that the *matrix* effective resistances in these lattices have the same scaling laws (see [8, Lemma 5]). Thus, 1-D, 2-D, and 3-D lattices with matrix-valued resistances have linear, logarithmic, and bounded scaling laws for the effective resistance, which is the result with $h = 1$. The case $h > 1$ follows from the application of Lemma 4. ■

The slowing down of the growth of the effective resistance as the dimension increases can be attributed to the fact that the number of paths between each pair of nodes is larger in higher dimensional lattices. The scaling laws for effective resistance in lattices and their fuzzes also have intimate connections to the change from recurrence to transience of random walks in lattices as the dimension changes from $d = 1, 2$ to $d \geq 3$ [16].

V. Proof of Theorem 1

We now prove Theorem 1 by using the tools that have been developed so far. The following terminology is needed for the proofs. For functions $g : \mathbb{R} \to \mathbb{R}^{k\times k}$ and $p : \mathbb{R} \to \mathbb{R}$, the notation $g(y) = \Theta(p(y))$ means that $g(y) = \Omega(p(y))$ and $g(y) = \mathcal{O}(p(y))$. The notations $\mathcal{O}(\,\cdot\,)$ and $\Omega(\,\cdot\,)$ are described in Section II.

Proof of Theorem 1: [Upper Bounds]: We start by establishing the upper bounds on the effective resistance for graphs that are dense in $\mathbb{R}^d$. Throughout the proof of the upper bounds, we will use $R^{\text{eff}}_{u,v}(\mathcal{G})$, for any graph $\mathcal{G}$, to denote the effective resistance between nodes u and v in the electrical network $(\mathcal{G}, P_{\max})$ with every edge of $\mathcal{G}$ having a generalized resistance of $P_{\max}$. From the electrical analogy theorem and monotonicity law (Theorems 4 and 5), we get

$$\Sigma_{u,o} \leq R^{\text{eff}}_{u,o}(\mathcal{G}).$$

To establish an upper bound on $\Sigma_{u,o}$, we will now establish an upper bound on the resistance $R^{\text{eff}}_{u,o}(\mathcal{G})$. To this effect, suppose that f is a dense drawing of G in $\mathbb{R}^d$. From dense embedding Theorem 3, we conclude that there exists a positive integer h such that the d-dimensional lattice $\mathbf{Z}_d$ can be embedded in the h-fuzz of $\mathcal{G}$. Moreover, Theorem 3 tells us that there exists $u_z, o_z \in \mathcal{V}_{\mathbf{Z}_d}$, a positive constant c, and an embedding $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$ of $\mathbf{Z}_d$ into $\mathcal{G}^{(h)}$, such that

$$d_{\mathcal{G}}(u, \eta(u_z)) \leq c, d_{\mathcal{G}}(o, \eta(o_z)) \leq c \tag{11}$$

$$d_{\mathbf{Z}_d}(u_z, o_z) < 4d + \frac{\sqrt{d}}{\gamma} d_f(u, o) \tag{12}$$

where γ is the maximum uncovered diameter of the f-drawing of $\mathcal{G}$. Note that $\eta(u_z), \eta(o_z) \in \mathcal{V}$. Consider the electrical network $(\mathcal{G}^{(h)}, P_{\max})$ formed by assigning to every edge of $G^{(h)}$ a resistance of $P_{\max}$. From the triangle inequality for effective resistances (Lemma 3)

$$R^{\text{eff}}_{u,o}\left(\mathcal{G}^{(h)}\right) \leq R^{\text{eff}}_{u,\eta(u_z)}\left(\mathcal{G}^{(h)}\right) + R^{\text{eff}}_{\eta(u_z),\eta(o_z)}\left(\mathcal{G}^{(h)}\right) + R^{\text{eff}}_{\eta(o_z),o}\left(\mathcal{G}^{(h)}\right). \tag{13}$$

For any two nodes $u, v \in \mathcal{V}$, application of the triangle inequality Lemma 3 to successive nodes on the shortest path joining u and v gives us $R^{\text{eff}}_{u,v}(\mathcal{G}^{(h)}) \leq d_{\mathcal{G}^{(h)}}(u, v) P_{\max} \leq$

$\frac{2}{h} d_{\mathcal{G}}(u,v) P_{\max}$. Using this bound in (13), and by using (11), we conclude that

$$R^{\text{eff}}_{u,o}\left(\mathcal{G}^{(h)}\right) \leq \frac{4c}{h} P_{\max} + R^{\text{eff}}_{\eta(u_z),\eta(o_z)}\left(\mathcal{G}^{(h)}\right). \tag{14}$$

Since $\mathcal{G}^{(h)} \supset \mathbf{Z}_d$, from Rayleigh's monotonicity law (Theorem 5), we obtain

$$R^{\text{eff}}_{\eta(u_z),\eta(o_z)}\left(\mathcal{G}^{(h)}\right) \leq R^{\text{eff}}_{u_z,o_z}(\mathbf{Z}_d).$$

When $\mathcal{G}$ is dense in, say, in $\mathbb{R}^2$, we have from Lemma 5 that

$$R^{\text{eff}}_{u_z,o_z}(\mathbf{Z}_2) = \Theta\left(\log d_{\mathbf{Z}_2}(u_z,o_z)\right)$$

which implies

$$R^{\text{eff}}_{\eta(u_z),\eta(o_z)}(\mathcal{G}^{(h)}) = \mathcal{O}\left(\log d_{\mathbf{Z}_2}(u_z,o_z)\right).$$

Combining this with (12) and (14), we get

$$R^{\text{eff}}_{u,o}\left(\mathcal{G}^{(h)}\right) = \mathcal{O}\left(\log d_f(u,o)\right).$$

From Lemma 4, we know that the effective resistance in $\mathcal{G}$ and its h-fuzz is of the same order, so that

$$R^{\text{eff}}_{u,o}(\mathcal{G}) = \Theta\left(R^{\text{eff}}_{u,o}(\mathcal{G}^{(h)})\right)$$

from which the desired result follows:

$$\Sigma_{u,o} \leq R^{\text{eff}}_{u,o}(\mathcal{G}) = \mathcal{O}(\log d_f(u,o)).$$

The statements of the upper bounds for 1-D and 3-D can be proved similarly. This concludes the proof of the upper bounds in Theorem 1.

[Lower Bounds]: Now we establish the lower bounds on the BLUE error covariance $\Sigma_{u,o}$ in a sparse graph. Throughout the proof of the lower bounds, for a graph $\mathcal{G}$, we will use $R^{\text{eff}}_{u,v}(\mathcal{G})$ to denote the effective resistance between nodes u and v in the electrical network $(\mathcal{G}, P_{\min})$ with every edge of $\mathcal{G}$ having a generalized resistance of $P_{\min}$. From the electrical analogy and Rayleigh's monotonicity law (Theorems 4 and 5), we get

$$\Sigma_{u,o} \geq R^{\text{eff}}_{u,o}(\mathcal{G}). \tag{15}$$

Therefore, to establish a lower bound on $\Sigma_{u,o}$, we proceed by establishing a lower bound on the resistance $R^{\text{eff}}_{u,o}(\mathcal{G})$. Since $\mathcal{G}$ is sparse in $\mathbb{R}^d$, it follows from Theorem 2 that there exists a positive integer h, such that $\mathcal{G} \subset \mathbf{Z}^{(h)}_d$. Let $\eta : \mathcal{V} \to \mathcal{V}_{\mathbf{Z}_d}$ be the embedding of $\mathcal{G}$ into $\mathbf{Z}^{(h)}_d$. Consider the generalized electrical network $(\mathbf{Z}^{(h)}_d, P_{\min})$ formed by assigning a generalized resistance of $P_{\min}$ to every edge of $\mathbf{Z}^{(h)}_d$. From Rayleigh's monotonicity law, we get

$$R^{\text{eff}}_{u,o}(\mathcal{G}) \geq R^{\text{eff}}_{u_z,o_z}\left(\mathbf{Z}^{(h)}_d\right) \tag{16}$$

where $u_z = \eta(u), o_z = \eta(o)$ refer to the nodes in $\mathbf{Z}^{(h)}_d$ that correspond to the nodes u, o in $\mathcal{G}$. When the graph is sparse in, say, $\mathbb{R}^2$, it follows from (16) and Lemma 5 that

$$\begin{aligned} R^{\text{eff}}_{u,o}(\mathcal{G}) &= \Omega\left(\log d_{\mathbf{Z}_2}(u_z,o_z)\right) \\ &= \Omega(\log d_f(u,o)) \end{aligned}$$

where the second statement follows from (6) in Theorem 2. Combining the above with (15), we get $\Sigma_{u,o} = \Omega(\log d_f(u,o))$, which proves the lower bound for graph that are sparse in $\mathbb{R}^2$. The statements for the lower bounds graphs that are sparse in $\mathbb{R}^1$ or $\mathbb{R}^3$ can be proved in an analogous manner. This concludes the proof of the theorem. ■

VI. Checking Denseness and Sparseness

To show that a graph is dense (or sparse) in a particular dimension, one has to find a drawing in that dimension with the appropriate properties. For sensor networks, sometimes the *natural drawing* of a deployed network is sufficient for this purpose. By the natural drawing of a sensor network, we mean the mapping from the nodes to their physical locations in the Euclidean space in which they are deployed. We can use this natural drawing to construct the following examples of dense and sparse graphs.

Proposition 1:

1) Deploy a countable number of nodes in $\mathbb{R}^d$ so that the maximum uncovered diameter γ of its natural drawing is finite, and allow every pair of nodes whose Euclidean distance is no larger than 2γ to have an edge between them. The resulting graph is weakly connected and dense in $\mathbb{R}^d$. Such a graph is also sparse in $\mathbb{R}^d$ if the nodes are placed such that every finite volume in $\mathbb{R}^d$ contains a finite number of nodes.
2) Consider an initial deployment of nodes on a square lattice in $\mathbb{R}^2$, for which a fraction of the nodes has subsequently failed. Suppose that the number of nodes that failed in any given region is bounded by a linear function of the area of the region, i.e., that there exist constants α and β such that, for every region of area A, the number of nodes that failed in that region is no larger than $\alpha A + \beta$. Assuming that $\alpha < \frac{1}{4(\beta+1)}$, there will be an infinite connected component among the remaining nodes, which is dense and sparse in 2-D. □

The interested reader may consult [6] for the proof of the proposition.

The first example in the proposition is that of a geometric graph that is obtained by placing a number of nodes in a region and specifying a range such that a pair of nodes have an edge between them if and only if the Euclidean distance between them is no more than the given range. The second example refers to a network in which some of the initially deployed nodes have failed, with the stipulation that in large areas, no more than a certain fraction of the node may fail. For example, $\beta = 5$ and $\alpha = 0.04$ satisfy the stated conditions. It can be shown that $\beta = 5$ and $\alpha = 0.04$ mean that in areas larger than 10×10, at most 4% of the nodes may fail.

To show that a graph is not dense (or not sparse) in a particular dimension is harder since one has to show that no drawing with the required properties exists. Typically, this can be done by showing that the existence of a dense (or sparse) drawing leads to a contradiction. An application of this technique leads to the following result.

Lemma 6:

1) The d-dimensional lattice $\mathbf{Z}_d$ is not sparse in $\mathbb{R}^{\underline{d}}$ for every $\underline{d} < d$, and it is not dense in $\mathbb{R}^{\overline{d}}$ for every $\overline{d} > d$.
2) A regular-degree[2] infinite tree is not dense or sparse in any dimension. □

The proof of the first statement of the lemma is provided in the Appendix. The second statement can be proved in an analogous manner.

We are now ready to prove Lemma 2.

Proof of Lemma 2: To prove the result by contradiction, suppose that a graph $\mathcal{G}$ is dense in $\mathbb{R}^d$ as well as sparse in $\mathbb{R}^{d'}$, where $d' < d$. It follows from Theorems 3 and 2 that there exist positive integers ℓ, p such that $\mathbf{Z}_d \subset \mathcal{G}^{(\ell)}$ and $\mathcal{G} \subset \mathbf{Z}_{d'}^{(p)}$. It is straightforward to verify the following facts:

1) for every pair of graphs $\mathcal{G}, \bar{\mathcal{G}}$ that do not have any parallel edges, $\mathcal{G} \subset \bar{\mathcal{G}} \Rightarrow \mathcal{G}^{(l)} \subset \bar{\mathcal{G}}^{(l)}$ for every positive integer l;
2) for an arbitrary graph $\mathcal{G}$ without parallel edges, and two positive integers ℓ, p, we have $(\mathcal{G}^{(p)})^{(\ell)} = \mathcal{G}^{(p\ell)}$.

It follows that $\mathbf{Z}_d \subset \mathbf{Z}_{d'}^{(\ell p)}$, which means, from sparse embedding Theorem 2, that a d-dimensional lattice is sparse in $\mathbb{R}^{d'}$. This is a contradiction because of Lemma 6, which completes the proof. ■

VII. Summary and Future Work

In a large number of sensor and *ad hoc* network applications, a number of node variables need to be estimated from measurements of the noisy differences between them. This estimation problem is naturally posed in terms of a graph.

We established a classification of graphs, namely, dense or sparse in $\mathbb{R}^d, 1 \leq d \leq 3$, that determines how the optimal linear unbiased estimation error of a node grows with its distance from the reference node. The notion of denseness/sparseness introduced in this paper is distinct from the usual notion based on the average degree. In fact, we illustrated through examples that node degree is a poor measure of how the estimation error scales with distance.

The bounds and the associated graph classification derived here can be used in performance analysis, design, and deployment of large networks. For example, if a sensor network is sparse in $\mathbb{R}$, then we know that the estimation error of a node will grow linearly with its distance from a reference. A large number of reference nodes will thus be needed for large networks that are sparse in $\mathbb{R}$. On the other hand, if one has control over the network deployment, then one should strive to obtain a network that is dense in $\mathbb{R}^d$ with d as large as possible. In the ideal case of $d = 3$, with a single reference node one can get bounded estimation error regardless of how large the network is.

There are several avenues for future research. The scaling laws described in this paper were derived for infinite measurement graphs. This is justified by the fact that the BLUE covariance $\Sigma_{u,o}$ of a node u in an infinite graph is very close to the obtained in a large finite subgraph that contains the nodes u and o sufficiently inside it [8]. However, to gain a better understanding of the "boundary" effects that can occur in finite graphs, an interesting research direction is to determine how large the BLUE error covariance can be as a function of the size of the graph, for nodes that are close to the edge of the graph. A connection between the notions introduced in this paper and those in coarse geometry might be useful in this regard. It can be shown that a graph $\mathcal{G}$ that is both sparse and dense in $\mathbb{R}^d$ is *coarsely equivalent* to $\mathbb{R}^d$, which intuitively means that $\mathcal{G}$ and $\mathbb{R}^d$ are the same in their large scale structure (see [24] for a precise definition of coarse equivalence). Certain coarse geometric notions that were originally defined for infinite graphs have been extended to finite graphs (see [25]). This connection between coarse geometry and denseness/sparseness might provide a way to extend the techniques used in this paper to finite graphs.

Although the dense and sparse classification does allow randomness in the structure of the graph, the effect of such randomness on the scaling laws for the error is not explicitly accounted for in the present work. A useful research direction would be the characterization of the estimation error covariances in graphs with random structure, such as random geometric graphs [26]. Another interesting avenue for future research is the investigation of estimation error growth in scale-free networks that do not satisfy the bounded degree assumption.

[2]A graph is called regular degree if the degree of every node in the graph is the same.

Appendix
Technical Proofs

Proof of Lemma 1: We prove that 1) implies 2) by contradiction. Assuming that 2) does not hold, we have that

$$\forall \alpha > 0 \qquad \forall \beta > 0 \qquad \exists \bar{u}, \quad \bar{v} \in \mathcal{V} \text{ such that } d_{\mathcal{G}}(\bar{u}, \bar{v}) > \alpha d_f(\bar{u}, \bar{v}) + \beta$$

or equivalently

$$\forall \alpha > 0 \qquad \forall \beta > 0 \qquad \exists \bar{u}, \bar{v} \in \mathcal{V}$$

such that

$$\frac{d_f(\bar{u}, \bar{v})}{d_{\mathcal{G}}(\bar{u}, \bar{v})} < \frac{1}{\alpha} - \frac{\beta}{\alpha d_{\mathcal{G}}(\bar{u}, \bar{v})}.$$

This means that for a given $\alpha > 0, \beta > 0$, the set

$$\left\{ \frac{d_f(u, v)}{d_{\mathcal{G}}(u, v)} : u, v \in \mathcal{V} \text{ and } d_{\mathcal{G}}(u, v) \geq \beta \right\}$$

contains at least the element

$$\frac{d_f(\bar{u}, \bar{v})}{d_{\mathcal{G}}(\bar{u}, \bar{v})} < \frac{1}{\alpha} - \frac{\beta}{\alpha d_{\mathcal{G}}(\bar{u}, \bar{v})} < \frac{1}{\alpha}$$

and therefore

$$\inf \left\{ \frac{d_f(u, v)}{d_{\mathcal{G}}(u, v)} : u, v \in \mathcal{V} \text{ and } d_{\mathcal{G}}(u, v) \geq \beta \right\} < \frac{1}{\alpha}.$$

Making $\beta \to \infty$, we obtain that

$$\rho = \lim_{\beta \to \infty} \inf \left\{ \frac{d_f(u, v)}{d_{\mathcal{G}}(u, v)} : u, v \in \mathcal{V} \text{ and } d_{\mathcal{G}}(u, v) \geq \beta \right\} < \frac{1}{\alpha}.$$

But since α can be arbitrarily large, the above actually implies that $\rho = 0$, which contradicts 1).

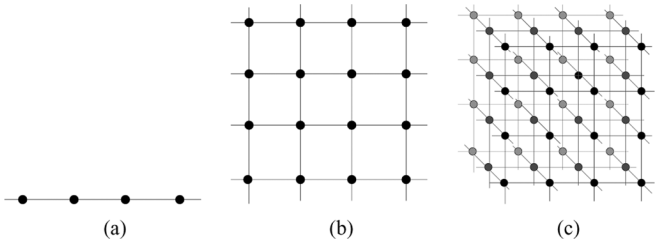

Fig. 4. Lattices: (a) 1-D lattice $\mathbf{Z}_1$, (b) 2-D lattice $\mathbf{Z}_2$, and (c) 3-D lattice $\mathbf{Z}_3$.

To prove that 2) implies 1), we note that when 2) holds, we conclude that for every pair of nodes $u, v \in \mathcal{V}$, for which $d_{\mathcal{G}}(u,v) \geq n$, we have that

$$\frac{d_f(u,v)}{d_{\mathcal{G}}(u,v)} \geq \frac{1}{\alpha} - \frac{\beta}{d_{\mathcal{G}}(u,v)} \geq \frac{1}{\alpha} - \frac{\beta}{n}, \qquad \forall u \neq v \in \mathcal{V}.$$

Therefore

$$\inf \left\{ \frac{d_f(u,v)}{d_{\mathcal{G}}(u,v)} : u, v \in \mathcal{V} \text{ and } d_{\mathcal{G}}(u,v) \geq n \right\} \geq \frac{1}{\alpha} - \frac{\beta}{n}.$$

As $n \to \infty$, the left-hand side converges to ρ and the right-hand side converges to $\frac{1}{\alpha} > 0$, from which 1) follows. ■

Proof of Theorem 3: In this proof, we will denote by $g : \mathcal{V}_{\mathbf{Z}_d} \to \mathbb{R}^d$ the natural drawing of the lattice $\mathbf{Z}_d$.

($\Rightarrow$) We have to prove that if $\mathcal{G}$ is dense in $\mathbb{R}^d$, conditions i) and ii) are satisfied. Since $\mathcal{G}$ is dense in $\mathbb{R}^d$, there is a drawing function $f : \mathcal{V} \to \mathbb{R}^d$ so that the f-drawing of $\mathcal{G}$ has a $\gamma < \infty$ and $\rho > 0$. Define a new drawing $f' : \mathcal{V} \to \mathbb{R}^d$ as

$$f'(u) = \frac{1}{\gamma} f(u) \qquad \forall u \in \mathcal{V}$$

so that the maximum uncovered diameter γ' of the f' drawing of $\mathcal{G}$ is 1. Note that f' is still a dense drawing of $\mathcal{G}$. Now we superimpose the natural g-drawing of $\mathbf{Z}_d$ on the f'-drawing of $\mathcal{G}$, and draw open balls of diameter 1 centered at the natural drawing $g(u_z)$ of every lattice node, denoted by $B(g(u_z), \frac{1}{2})$. Fig. 5 shows an example in $\mathbb{R}^2$. Since $\gamma' = 1$, it follows from the definition of denseness that in every one of those balls, there is at least one node $u \in \mathcal{V}$. To construct the embedding, we associate each node of the lattice to a node of $\mathcal{G}$ whose drawing appears inside the ball centered around the lattice node. This defines an injective function $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$. Consider two nodes of the lattice $u_z, v_z \in \mathcal{V}_{\mathbf{Z}_d}$ that have an edge between them. Let $\bar{u} := \eta(u_z), \bar{v} := \eta(v_z)$. Since $f'(\bar{u})$ and $f'(\bar{v})$ belong to adjacent balls of unit diameter (see Fig. 5)

$$d_{f'}(\bar{u}, \bar{v}) = \|f'(\bar{u}) - f'(\bar{v})\| \leq 2.$$

Since f' is a dense drawing in $\mathbb{R}^d$ with $\gamma' = 1$, it follows from Lemma 1 that $d_{\mathcal{G}}(\bar{u}, \bar{v}) \leq 2\alpha + \beta$, for some positive constants α and β. Define $h := \lceil 2\alpha + \beta \rceil$. Then $\bar{u}$ and $\bar{v}$ will have an edge between them in the h-fuzz $\mathcal{G}^{(h)}$. So $\mathcal{G}^{(h)} \supset \mathbf{Z}_d$, and we have the desired result that denseness implies i).

To show that denseness implies ii), first note that if $u \in \eta(\mathcal{V}_{\mathbf{Z}_d})$, then ii) is trivially true (choose $\bar{u} := u$), so only nodes in $\mathcal{V} \backslash \eta(\mathcal{V}_{\mathbf{Z}_d})$ are interesting. For every $u \in \mathcal{V}$, find $u_z \in \mathcal{V}_{\mathbf{Z}_d}$ as

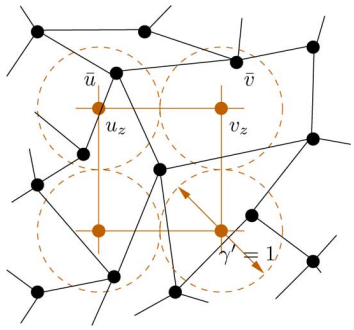

Fig. 5. Superimposing a 2-D lattice (gray) on a 2-D dense graph (black).

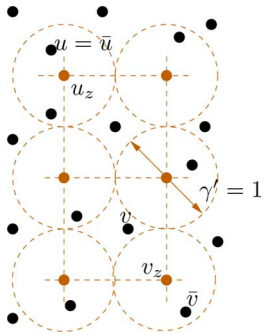

Fig. 6. Natural drawing of the 2-D lattice (gray) superimposed on the f' drawing of $\mathcal{G}$. Edges are not shown to prevent clutter. In this example, $u = \bar{u}$ but $v \neq \bar{v}$.

the node in the lattice such that the ball of unit diameter drawn around u_z is closest to u. That is, find $u_z \in \mathcal{V}_{\mathbf{Z}_d}$ such that

$$u_z = \arg \min_{u_z' \in \mathcal{V}_{\mathbf{Z}_d}} \operatorname{dist}(f'(u), B(g(u_z'), 1/2)) \qquad (17)$$

where $\operatorname{dist}(x, A)$ between a point $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$ is defined as

$$\operatorname{dist}(x, A) = \inf_{y \in A} \|x - y\|.$$

There are only 2^d balls one needs to check to determine the minimum in (17), so u_z exists, though it may not be unique. If there are multiple minima in (17), pick any one. This procedure defines an onto map $\xi : \mathcal{V} \to \mathcal{V}_{\mathbf{Z}_d}$. Let $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$ be the embedding of $\mathbf{Z}_d$ into $\mathcal{G}^{(h)}$ as described earlier in this proof. Define $\psi : \mathcal{V} \to \mathcal{V}$ as $\psi := (\eta \circ \xi)$. We will now show that, for every $u \in \mathcal{V}$, the node $\psi(u) \in \mathcal{V}$, which has a corresponding node in the lattice, is within a uniformly bounded graphical distance of u. Since $f'(u)$ either lies in the ball centered at $g(u_z)$ or in the gaps between that ball and the neighboring balls $\|f'(u) - g(u_z)\| < \sqrt{d}$. Therefore

$$\begin{aligned} d_{f'}(u, \psi(u)) &\leq \|f'(u) - g(u_z)\| + \|g(u_z) - f'(\psi(u))\| \\ &< \sqrt{d} + \frac{1}{2} \leq \frac{3}{2}\sqrt{d} \end{aligned} \qquad (18)$$

where we have used the fact that $f'(\psi(u)) \in B(g(u_z), \frac{1}{2})$. From Lemma 1 and the denseness of the f-drawing of $\mathcal{G}$, we get

$$\begin{aligned} d_{\mathcal{G}}(u, \psi(u)) &\le \alpha d_f(u, \psi(u)) + \beta \\ &= \alpha\gamma d_{f'}(u, \psi(u)) + \beta \\ &< \frac{3}{2}\alpha\gamma\sqrt{d} + \beta. \end{aligned}$$

Define

$$c := \left\lceil \frac{3}{2}\alpha\gamma\sqrt{d} + \beta \right\rceil \tag{19}$$

which is a constant independent of u and v. Then, for every $u \in \mathcal{V}$, there exists a $\bar{u} := \psi(u) \in \eta(\mathcal{V}_{\mathbf{Z}_d}) \subset \mathcal{V}$ such that $d_{\mathcal{G}}(u, \bar{u}) < c$, which is the desired condition ii).

($\Leftarrow$) We have to prove that if i) and ii) are satisfied, then $\mathcal{G}$ is dense in $\mathbb{R}^d$. We will construct a drawing f of $\mathcal{G}$ in $\mathbb{R}^d$ with the following procedure and then prove that it is a dense drawing. Since $\mathbf{Z} \subset \mathcal{G}^{(h)}$, there is an injective map $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$ such that $\eta(\mathcal{V}_{\mathbf{Z}_d}) \subset \mathcal{V}$. Pick a node u in $\mathcal{V}$ that has not been drawn yet. By ii), there exist a positive constant c and a node $u_z \in \mathcal{V}_{\mathbf{Z}_d}$ such that $\bar{u} := \eta(u_z) \in \mathcal{V}$ and $d_{\mathcal{G}}(u, \bar{u}) < c$. If $\bar{u}$ has not been drawn yet, then draw it the location of its corresponding lattice node, i.e.,

$$f(\bar{u}) = g(u_z). \tag{20}$$

A little thought will reveal that if $\bar{u}$ has been drawn already, as long as the drawing procedure outlined so far is followed, it must have been drawn on the lattice location $g(u_z)$, so (20) holds. Once $\bar{u}$ is drawn, we draw u in the following way. In case $\bar{u} = u$, drawing of u is determined by the drawing of $\bar{u}$. If $u \neq \bar{u}$, draw u by choosing a random location inside an open ball of diameter 1 with the center at $f(\bar{u})$. To show that a drawing obtained this way is dense, first note that the largest uncovered diameter $\gamma < 2$ since a subset of the nodes of $\mathcal{V}$ occupies the lattice node positions. Pick any two nodes $u, v \in \mathcal{V}$. Again, from ii), we know that there exists $\bar{u}, \bar{v} \in \eta(\mathcal{V}_{\mathbf{Z}_d}) \subset \mathcal{V}$ such that $d_{\mathcal{G}}(u, \bar{u}) \le c$ and $d_{\mathcal{G}}(v, \bar{v}) \le c$ for some positive constant c. Therefore

$$\begin{aligned} d_{\mathcal{G}}(u, v) &\le d_{\mathcal{G}}(u, \bar{u}) + d_{\mathcal{G}}(\bar{u}, \bar{v}) + d_{\mathcal{G}}(\bar{v}, v) \\ &\le 2c + h\, d_{\mathcal{G}^{(h)}}(\bar{u}, \bar{v}). \end{aligned}$$

Since $\mathbf{Z}_d \subset \mathcal{G}^{(h)}$

$$\begin{aligned} d_{\mathcal{G}^{(h)}}(\bar{u}, \bar{v}) &\le d_{\mathbf{Z}_d}(\eta^{-1}(\bar{u}), \eta^{-1}(\bar{v})) \\ &= \|g(u_z) - g(v_z)\|_1 \end{aligned}$$

where $\|\cdot\|_1$ denotes the vector 1-norm

$$\begin{aligned} &\le \sqrt{d}\|g(u_z) - g(v_z)\| \\ &= \sqrt{d}\|f(\bar{u}) - f(\bar{v}))\| \quad \text{[from (20)]} \\ &= \sqrt{d}\, d_f(\bar{u}, \bar{v}). \end{aligned}$$

Because of the way the drawing f is constructed, we have $d_f(u, \bar{u}) \le 1$, which implies $d_f(\bar{u}, \bar{u}) \le d_f(\bar{u}, u) + d_f(u, v) + d_f(v, \bar{v}) = d_f(u, v) + 2$. So we have

$$\begin{aligned} d_{\mathcal{G}}(u, v) &\le 2c + h\sqrt{d}\,(d_f(u, v) + 2) \\ &= 2(c + h\sqrt{d}) + h\sqrt{d} d_f(u, v). \end{aligned}$$

From Lemma 1, we see that the asymptotic distance ratio $\rho > 0$ for the f-drawing of $\mathcal{G}$, which establishes that f is a dense drawing of $\mathcal{G}$ in $\mathbb{R}^d$. It follows that $\mathcal{G}$ is dense in $\mathbb{R}^d$.

To prove the relationship (7) for any dense drawing f, consider again the scaled drawing f' defined as $f' = f/\gamma$, so that the maximum uncovered diameter of f' is 1. Since $\mathcal{G}$ is dense in $\mathbb{R}^d$, $\mathbf{Z}_d$ can be embedded in $\mathcal{G}^{(h)}$ with an embedding $\eta : \mathcal{V}_{\mathbf{Z}_d} \to \mathcal{V}$. We choose the embedding η as described in the first part of the proof. For every $u \in \mathcal{V}$, call $u_z := \xi(u)$, where $\xi : \mathcal{V} \to \mathcal{V}_{\mathbf{Z}_d}$ was defined earlier in this proof for the f' dense drawing of $\mathcal{G}$. Now consider two arbitrary nodes $u, v \in \mathcal{V}$ and let $u_z := \xi(u), v_z := \xi(v)$ (see Fig. 6). It was shown earlier in this proof that for every pair of nodes $u, v \in \mathcal{V}$, we have $d_{\mathcal{G}}(u, \eta(u_z)) < c$ and $d_{\mathcal{G}}(v, \eta(v_z)) < c$, where c is defined in (19).

Now

$$\begin{aligned} d_{\mathbf{Z}_d}(u_z, v_z) &= \|g(u_z) - g(v_z)\|_1 \\ &\le \sqrt{d}\|g(u_z) - g(v_z)\| \end{aligned}$$

and

$$\begin{aligned} \|g(u_z) - g(v_z)\| \le{}& \|g(u_z) - f'(\bar{u})\| + \|f'(\bar{u}) - f'(u)\| \\ &+ \|f'(u) - f'(v)\| + \|f'(v) - f'(\bar{v})\| \\ &+ \|f'(\bar{v}) - g(v_z)\|. \end{aligned}$$

We know that $\|g(u_z) - f'(\bar{u})\| \le \frac{1}{2} \le \frac{\sqrt{d}}{2}$ since $f'(\bar{u}) \in B(g(u_z), \frac{1}{2})$, and $\|f'(u) - f'(\bar{u})\| < \frac{3}{2}\sqrt{d}$ from (18). Using these in the above, we get

$$\begin{aligned} \|g(u_z) - g(v_z)\| &\le 4\sqrt{d} + d_{f'}(u, v) \\ &\Rightarrow d_{\mathbf{Z}_d}(u_z, v_z) \le 4d + \frac{\sqrt{d}}{\gamma} d_f(u, v) \end{aligned}$$

which is the desired result. ∎

Proof of Lemma 6: We only provide the proof that the 2-D lattice is not sparse in $\mathbb{R}$ and is not dense in $\mathbb{R}^3$. The general case for arbitrary dimensions is analogous.

To prove by contradiction the lack of denseness, assume that there exists a dense drawing f of $\mathbf{Z}_2$ in $\mathbb{R}^3$, with associated $\gamma < \infty$ and $\rho > 0$. Fix the origin of $\mathbb{R}^3$ at $f(u)$ for an arbitrary node u in the lattice $\mathbf{Z}_2$. For an arbitrary $D > 0$, the volume of the sphere in $\mathbb{R}^3$ centered at the origin with diameter D, denoted by $\mathcal{B}^3(0, D)$, is $\Omega(D^3)$. Therefore, the number of nodes of $\mathbf{Z}_2$ drawn inside $\mathcal{B}^3(0, D)$ is $\Omega((\frac{D}{\delta})^3) = \Omega(D^3)$. It is straightforward to show that for any set of n distinct nodes in the lattice $\mathbf{Z}_2$, the maximum graphical distance between any two nodes in the set is $\Omega(\sqrt{n})$. Therefore, the maximum graphical distance between the nodes in $\mathcal{B}^3(0, D)$ is $\Omega(D^{\frac{3}{2}})$. The maximum Euclidean distance between any two nodes drawn inside

the sphere $\mathcal{B}^3(0, D)$ under the f-drawing is at most D, and since f is a dense drawing, it follows from Lemma 1 that for every pair of nodes u, v in $\mathbf{Z}_2$ such that $f(u), f(v) \in \mathcal{B}^3(0, D)$, we have $d_G(u, v) \leq aD + b$. Therefore, the maximum graphical distance between pairs of nodes whose drawing falls inside $\mathcal{B}^3(0, D)$ is $\mathcal{O}(D)$, as well as $\Omega(D^{\frac{3}{2}})$, which is a contradiction for sufficiently large D. Hence, no dense drawing of $\mathbf{Z}_2$ in $\mathbb{R}^3$ is possible.

To show $\mathbf{Z}_2$ is not sparse in $\mathbb{R}$, assume that there exists a civilized drawing of $\mathbf{Z}_2$ in $\mathbb{R}$ with $s > 0$ and $r < \infty$, where r and s are constants. Consider a subgraph $\mathbf{Z}_{2(n)}$ of $\mathbf{Z}_2$ that consists all nodes within a Euclidean distance n from the origin. The total number of nodes in this finite subgraph is $\Omega(n^2)$. The length of the interval L in which the nodes of this subgraph are located in the sparse 1-D drawing of $\mathbf{Z}_2$ is clearly $L = \Omega(sn^2)$. Since the maximum graphical distance between any two nodes in the subgraph $\mathbf{Z}_{2(n)}$ is n by construction, the maximum connected range in the 1-D drawing must be at least $r \geq \frac{L}{n} = \Omega(sn)$. Since this must be true for every n, r cannot be a finite constant. Thus, no civilized drawing of $\mathbf{Z}_2$ in $\mathbb{R}$ exists. ■

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