Building Thermal Model Reduction via Aggregation of States

Kun Deng, Prabir Barooah, Prashant G. Mehta, and Sean P. Meyn

***Abstract*— This paper proposes an aggregation-based model reduction method for thermal models of large buildings. Using an electric analogy, the baseline thermal model is represented as an RC-network. The proposed model reduction methodology is used to obtain a simpler (with fewer states) multi-scale representation of this network. The methodology preserves the electrical analogy and retains the physical intuition during the model reduction process. The theoretical results are illustrated with the aid of examples.**

I. Introduction

A recent NREL (National Renewable Energy Laboratory) study identifies the "lack of innovative controls and monitoring systems" as one of the principal bottlenecks in achieving high energy efficiency in buildings [1]. Real time monitoring and control is therefore likely to play a more significant role in operating the HVAC (Heating, Ventilation, and Air Conditioning) equipment in commercial buildings than it has played so far. To be effective, control and monitoring systems must be guided by mathematical models of thermal transport in large buildings.

The dynamics of temperature evolution in a building is one of the most important aspects of the overall building dynamics. The complexity in the dynamics of temperature evolution comes from the thermal interaction among rooms (and the outside). This interaction can be either through conduction through the walls, or through convective air exchange among rooms. In this paper, we focus on modeling and model reduction of the thermal interaction among rooms of a building. The thermal effects of the cooling loads and conditioned/exhaust air are ignored here but can be specified as an exogenous input to the model.

An extensive literature exists on modeling the conductive interaction between *two* spaces through the wall separating them. The most popular modeling framework consists of using resistors and capacitors to model this interaction [2]–[4]. Work by Gouda *et. al.* showed that a second-order RC-network model with 3 resistors and 2 capacitors is sufficient to capture the conductive dynamic interaction between two spaces connected through a single wall [5]. Thus it is possible to model the conductive interaction in a mutli-room building by using such simpler RC-networks as building blocks. In this formulation, the building is represented by a *graph* with nodes and edges. A node may represent a physical zone (e.g., a room, a hallway, or "the outside"), or some point inside a wall. Edges represent pathways for conductive heat transport. The resulting model of the building consists of a large electrical network of resistors and capacitors. Temperature of a room is analogous to the voltage of the corresponding node, and the net flow of heat into a room is analogous to the net current into the corresponding node.

The fundamental problem with such models, which we will call *network models*, is that they quickly explode in complexity. For example, consider a 4-room building shown in Fig. 1 (a). A network model of this needs 37 nodes and 51 edges (see Fig. 1 (b) and also see Section V-B for detailed descriptions). For a realistic large commercial building, the number of nodes and edges will be on the order of several thousands. For monitoring and control, model reduction thus becomes necessary.

Since the model structure is linear, many approaches to model reduction that exist in literature, are potentially applicable. These include balanced model reduction approaches, approaches based on frequency-domain approximation, moment matching methods, projection-based methods, SVD decomposition and Krylov subspace based techniques, etc [6]. The focus of this paper is instead on *aggregation-based approaches* that preserve the electrical analogy interpretation of the original model. The goal is to obtain the super-nodes based upon an aggregation technique, and find the super-capacitance for each super-node and super-resistance for each edge between two adjacent super-nodes. The resulting reduced-order model is used to describe the thermal dynamics of the aggregated building.

The reason for choosing the aggregation based methodology is two-fold. One, such a methodology is expected to reveal the multiple time-scales that are inherent in any building model. Understanding and modeling of such scales is important for a hierarchical control architecture of HVAC where the scheduling of chillers is done on a slow time-scale and individual room temperature is controlled using PI loops on a much faster time-scale. The other reason is that zone-based models are standard in the HVAC community [7], [8]. Multi-scale models that retain the physical intuition are more likely to be incorporated as part of standard practice.

The approach proposed in this paper is based on model reduction of Markov chains that has recently been developed by the authors of this paper [9]. In this work, Kullback-Leibler (KL) divergence rate (or relative entropy rate) is proposed to reduce regular Markov chains via aggregation. The idea of this paper is to connect the thermal models to Markov chains and replicate the model reduction procedure

Kun Deng, Prashant G. Mehta, and Sean P. Meyn are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 West Main Street, Urbana, IL 61801. Email: kundeng2@illinois.edu, mehtapg@illinois.edu, and meyn@illinois.edu

Prabir Barooah is with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL 32611. Email: pbarooahg@ufl.edu

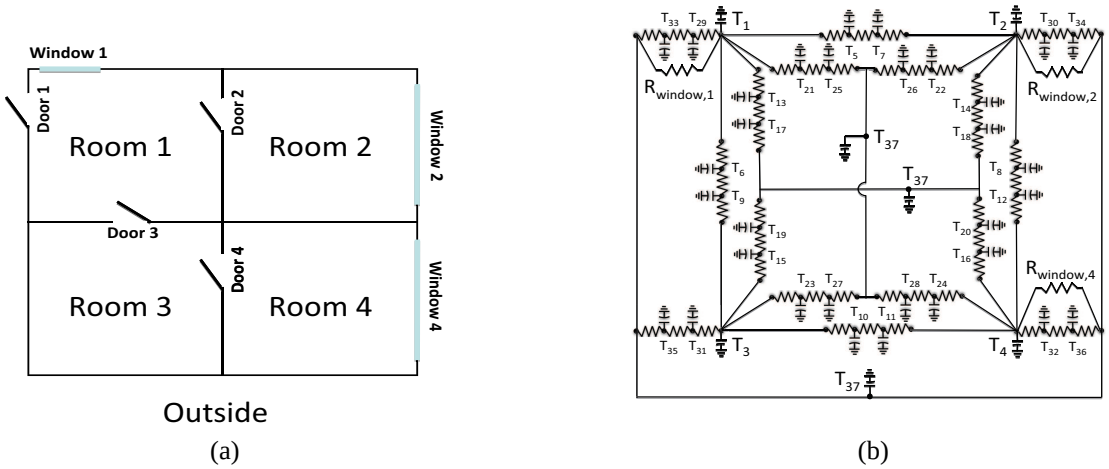


Fig. 1. (a) Layout of a 4-room building and (b) the RC-network representation of the same building.

on the thermal models.

The outline of this paper is as follows. In Section II, the thermal model is formally defined and its Markov chain representation is presented. In Section III, the KL divergence rate and the model reduction method from [9] is briefly reviewed. In Section IV, the methodology is applied to reduce the thermal models. In Section V, two examples are presented to illustrate the theoretical results. The conclusions appear in Section VI.

II. Building Thermal Model

A. RC-network representation

The building topology is determined by an *undirected weighted graph* $G = (\mathcal{N}, \mathcal{E}, \mathcal{W})$, where $\mathcal{N} := \{1, 2, \ldots, n\}$ denotes the set of *nodes* of the graph, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ denotes the set of *edges* between two adjacent nodes, $\mathcal{R} := \{R_{ij} \in \mathbb{R}_+ : (i, j) \in \mathcal{E}\}$ denotes the set of *resistances* for the edges of G. The graph is undirected, i.e., the pair (i, j) and (j, i) denote the same edge. Each node $i \in \mathcal{N}$ is assigned a temperature T_i and a capacitance C_i. The resistance R_{ij} on edge (i, j) satisfies $R_{ij} = R_{ji}$.

Since the thermal model is an RC-network, its dynamics is described by a system of coupled first order linear differential equations of the form $C_i dT_i/dt = q_i$, where q_i is the net heat flow into node i through the resistive elements connected to it. The net heat flow into node i from node j through the edge (i, j) is simply modeled as $(T_j - T_i)/R_{ij}$. This system of equations can be expressed in the state-space form:

$$\frac{dT}{dt}(t) = AT(t), \tag{1}$$

where the column vector $T(t) = [T_1(t), T_2(t), \ldots, T_n(t)]^T$ denotes the temperatures at time t, and the entries of the *transition-rate matrix* $A = (A_{ij}, i, j \in \mathcal{N})$ are given by

$$\begin{cases} A_{ij} = 0, & \text{if } i \neq j,\ (i, j) \notin \mathcal{E}; \\ A_{ij} = 1/(C_i R_{ij}), & \text{if } i \neq j,\ (i, j) \in \mathcal{E}; \\ A_{ii} = -\sum_{j \neq i} A_{ij}, & \text{if } i = j,\ (i, j) \in \mathcal{E}. \end{cases} \tag{2}$$

The initial temperature is denoted by $T(0)$.

Remark 1 (Conservative Property) *The row sums of A matrix are all zeros, its diagonal entries are all negative, and its non-diagonal entries are all non-negative. The quantity $V(t) = \sum_{i \in \mathcal{N}} C_i T_i(t)$ denotes the* total heat *of the building thermal model (1) at time t. One can justify that*

$$\frac{dV}{dt}(t) = \sum_{i \in \mathcal{N}} C_i \frac{dT_i}{dt}(t) = 0.$$

This means $V(t) = V(0)$ is invariant with respect to the time t. We denote $V := V(0) = \sum_{i \in \mathcal{N}} C_i T_i(0)$ as the invariant quantity *of the building thermal model.*

B. Goal of model reduction

For model reduction, the idea is to aggregate the node set $\mathcal{N} = \{1, 2, \ldots, n\}$ into a smaller super-node set $\mathcal{M} = \{1, 2, \ldots, m\}$ where $m \leq n$. The relationship between $\mathcal{N}$ and $\mathcal{M}$ is described by a *partition function* ϕ:

Definition 1 *Let $\mathcal{N} = \{1, 2, \ldots, n\}$ and $\mathcal{M} = \{1, 2, \ldots, m\}$ be two finite sets with $m \leq n$. A* partition function *$\phi : \mathcal{N} \mapsto \mathcal{M}$ is a surjective function from $\mathcal{N}$ onto $\mathcal{M}$. For $k \in \mathcal{M}$, $\phi^{-1}(k)$ denotes the k^{th} group of nodes in $\mathcal{N}$.*

For each super-node $k \in \mathcal{M}$, we introduce the following notations: super-temperature $\bar{T}_k$, super-capacitance $\bar{C}_k$ and super-resistance $\bar{R}_{kl}$. Then a reduced-order model is used to describe the thermal dynamics for the super-nodes

$$\frac{d\bar{T}^{(\phi)}}{dt}(t) = \bar{A}(\phi)\bar{T}^{(\phi)}(t), \tag{3}$$

where $\bar{T}^{(\phi)}(t) = [\bar{T}_1^{(\phi)}(t), \bar{T}_2^{(\phi)}(t), \ldots, \bar{T}_m^{(\phi)}(t)]^T$ denotes the super-temperature vector with the partition function ϕ at time t, and $\bar{A}(\phi)$ denotes the $m \times m$ super-transition-rate matrix with the partition function ϕ.

The goal of the model reduction is to find the optimal partition function and the optimal reduced-order building thermal model (defined by (3)) such that the reduced-order

model is the best approximation of the original model (defined by (1)), i.e., the modeling error is minimized between the original and the reduced-order models.

C. Representation via a regular Markov chain

Based upon the conservative property of the thermal model, we define the *heat distribution* as a row vector, denoted by $f(t) = [f_1(t), f_2(t), \ldots, f_n(t)]$, where

$$f_i(t) = \frac{C_i}{V} T_i(t), \quad i \in \mathcal{N} \tag{4}$$

and V is the invariant quantity defined in Remark 1.

For $i \in \mathcal{N}$, by differentiating f_i and using (1), we have

$$\frac{df_i}{dt} = \frac{C_i}{V}\frac{dT_i}{dt} = \sum_{j\in\mathcal{N}} A_{ij}\frac{C_i}{V} T_j. \tag{5}$$

By substituting (2) in (5), we have

$$\frac{df_i}{dt} = A_{ii}\frac{C_i}{V}T_i + \sum_{j\neq i}\frac{1}{C_j R_{ji}}\frac{C_j}{V}T_j = \sum_{j\in\mathcal{N}} f_j A_{ji},$$

where we use the fact that $R_{ij} = R_{ji}$ in derivation.

The thermal dynamics of the building is now analogous to a time-homogeneous Markov chain defined on the finite state-space $\mathcal{N}$ with the transition matrix $P(t) = e^{At}$.

Assumption 1 *All Markov chains considered in this paper are* regular *chains, i.e., they are irreducible and aperiodic.*

For a regular finite Markov chain, the *stationary distribution* exists, which is denoted by by a row vector

$$\pi := \lim_{t\to\infty} f(t).$$

It is also the *invariant measure* of the Markov chain, i.e.,

$$\pi P(t) = \pi, \quad \forall t \geq 0. \tag{6}$$

Substituting $P(t) = e^{At}$ into (6), we have $\pi A = 0$. Then the stationary distribution can be explicitly computed for the building thermal model

$$\pi_i = \frac{C_i}{\sum_{j\in\mathcal{N}} C_j}, \quad i \in \mathcal{N}. \tag{7}$$

III. Aggregation of a Regular Markov Chain

In this section, we summarize the main results of our recent work [9], in which KL divergence rate is used to aggregate a regular discrete-time Markov chain. The notation (π, P) is used to denote a regular Markov chain with the transition matrix P and the stationary distribution π.

A. KL divergence rate for Markov chains

For model reduction problems, it is of interest to compare two Markov chains (π, P) and (ϖ, Q) defined on different state spaces $\mathcal{N}$ and $\mathcal{M}$, respectively. Let ϕ denote the partition function from $\mathcal{N}$ to $\mathcal{M}$ and π be the invariant measure on $\mathcal{N}$ such that $\pi P = \pi$. The KL divergence rate is defined for two Markov chains on *different state spaces* as:

$$R(P \parallel \widehat{Q}^{(\pi)}(\phi)) = \sum_{i,j\in\mathcal{N}} \pi_i P_{ij} \log\left(\frac{P_{ij}}{\widehat{Q}^{(\pi)}(\phi)_{ij}}\right), \tag{8}$$

where

$$\widehat{Q}^{(\pi)}_{ij}(\phi) = \frac{\pi_j}{\sum_{k\in\psi(j)}\pi_k} Q_{\phi(i)\phi(j)}, \quad i,j \in \mathcal{N}$$

where $\psi(j) = \phi^{-1}\circ\phi(j) \subset \mathcal{N}$ denotes the set of states belonging to the same group as the j^{th} state.

B. Optimal aggregation problem

Let (π, P) be a given regular Markov chain on $\mathcal{N}$. The *m-partition problem*, is to find the partition function $\phi : \mathcal{N} \mapsto \mathcal{M}$ and the optimal aggregated Markov chain (ϖ, Q) such that $R^{(\phi)}(P \parallel Q)$ is minimized:

$$\begin{array}{ll} \min\limits_{\phi, Q} & R^{(\phi)}(P \parallel Q) \\ \text{s.t.} & \sum_{l\in\mathcal{M}} Q_{kl} = 1, \quad k \in \mathcal{M} \\ & Q_{kl} \geq 0, \quad k,l \in \mathcal{M} \end{array}$$

where $R^{(\phi)}(P \parallel Q) = R(P \parallel \widehat{Q}^{(\pi)}(\phi))$ and constraints arise due to stochastic property of the Markov transition matrix.

As shown in Theorem 3 of [9], for a fixed (say an optimal) partition function ϕ, the optimal aggregated Markov chain $(\varpi(\phi), Q(\phi))$ can be easily obtained as a function of ϕ

$$Q_{kl}(\phi) = \frac{\sum_{i\in\phi^{-1}(k)}\sum_{j\in\phi^{-1}(l)}\pi_i P_{ij}}{\sum_{i\in\phi^{-1}(k)}\pi_i}, \quad k,l \in \mathcal{M}. \tag{9}$$

The stationary distribution of $Q(\phi)$ is given by

$$\varpi_k(\phi) = \sum_{i\in\phi^{-1}(k)} \pi_i, \quad k \in \mathcal{M}. \tag{10}$$

Then the m-partition problem becomes to obtain *only* the optimal partition function ϕ^* such that

$$\phi^* \in \arg\min_{\phi:\mathcal{N}\mapsto\mathcal{M}} R^{(\phi)}(P \parallel Q(\phi)), \tag{11}$$

where $Q(\phi)$ is the optimal aggregated Markov transition matrix (9) with the partition function ϕ.

In [9], a spectral partitioning algorithm is proposed to obtain a suboptimal solution of (11): For bi-partition problem, ϕ^* is given by the sign-structure of the second eigenvector of the symmetric matrix $\hat{P} = \frac{1}{2}(\Pi^{\frac{1}{2}} P \Pi^{-\frac{1}{2}} + \Pi^{-\frac{1}{2}} P^T \Pi^{\frac{1}{2}})$, where $\Pi = \text{diag}(\pi)$. The sub-optimal solution of the multi-partition problem is obtained via recursive application of the bi-partition algorithm.

IV. Thermal Interpretation of Aggregation

In this section, we apply the aggregation methodology for Markov chains to obtain a reduced-order model for the building thermal model (1).

A. Optimal reduced-order model

By letting $t \to 0$, we obtain the Markov transition matrix for the full-order building thermal model (1) defined on $\mathcal{N}$:

$$P(t) = e^{At} = I + At + O(t^2), \tag{12}$$

where I is the $n \times n$ identity matrix and A is the transition-rate matrix defined in (2). The stationary distribution of $P(t)$ is π, which can be explicitly obtained in terms of the capacitances of building thermal system (see (7)).

Similarly, we have the aggregated Markov transition matrix for the reduced-order model (3) defined on $\mathcal{M}$:

$$Q(t) = e^{\bar{A}t} = I + \bar{A}t + O(t^2), \tag{13}$$

where I is the $m \times m$ identity matrix and $\bar{A}$ is the super-transition-rate matrix. The stationary distribution of $Q(t)$ is denoted by ϖ, which can be obtained in terms of the super-capacitances

$$\varpi_k = \frac{\bar{C}_k}{\sum_{l\in\mathcal{M}} \bar{C}_l}, \quad k \in \mathcal{M}. \tag{14}$$

Substituting (12) into (9), we obtain the formula for the optimal aggregated Markov transition matrix for a fixed partition function ϕ, for $k, l \in \mathcal{M}$:

$$\begin{aligned} Q_{kl}(t,\phi) &= \frac{\sum_{i\in\phi^{-1}(k)} \sum_{j\in\phi^{-1}(l)} \pi_i(\mathbb{1}_{\{i=j\}} + A_{ij}t + O(t^2))}{\sum_{i\in\phi^{-1}(k)} \pi_i}, \\ &= \mathbb{1}_{\{k=l\}} + \frac{\sum_{i\in\phi^{-1}(k)} \sum_{j\in\phi^{-1}(l)} \pi_i A_{ij}}{\sum_{i\in\phi^{-1}(k)} \pi_i} t + O(t^2). \end{aligned} \tag{15}$$

By comparing (13) and (15), we obtain the optimal super-transition-rate matrix with the partition function ϕ:

$$\bar{A}_{kl}(\phi) = \frac{\sum_{i\in\phi^{-1}(k)} \sum_{j\in\phi^{-1}(l)} \pi_i A_{ij}}{\sum_{i\in\phi^{-1}(k)} \pi_i}, \quad k, l \in \mathcal{M}. \tag{16}$$

By substituting (7) into (10), we obtain the stationary distribution of the optimal aggregated Markov chain

$$\varpi_k(\phi) = \sum_{i\in\phi^{-1}(k)} \pi_i = \frac{\sum_{i\in\phi^{-1}(k)} C_i}{\sum_{l\in\mathcal{M}} \sum_{j\in\phi^{-1}(l)} C_j}. \tag{17}$$

By comparing (14) and (17), we obtain the formulae for super-capacitances

$$\bar{C}_k(\phi) = \sum_{i\in\phi^{-1}(k)} C_i, \quad k \in \mathcal{M}. \tag{18}$$

Using (16) and (18), we obtain the formulae for super-resistances

$$\bar{R}_{kl}(\phi) = \frac{1}{\bar{C}_k(\phi)\bar{A}_{kl}(\phi)}, \quad k \neq l \in \mathcal{M}.$$

Thus, after the aggregation of states/nodes, the reduced-order model is also an RC-network for any fixed partition function ϕ.

Comparing the full-order model and the optimal reduced-order model with $\bar{A}(\phi)$ given in (16), we find that the relationship between the super-temperature vector and the temperature vector is defined by a linear transformation:

$$\bar{T}_k^{(\phi)}(t) = \sum_{i\in\phi^{-1}(k)} (C_i/\bar{C}_k) T_i(t), \quad \forall t \geq 0, k \in \mathcal{M}. \tag{19}$$

Since the row sums of $\bar{A}(\phi)$ are also all zeros, then the optimal reduced thermal model is also conservative. The invariant quantity for the reduced thermal model is denoted by

$$\bar{V}(\phi) := \sum_{k\in\mathcal{M}} \bar{C}_k \bar{T}_k^{(\phi)}(0). \tag{20}$$

TABLE I

NUMERICAL VALUES FOR THE FULL-ORDER MODEL

$C_1 = 0.1$	$C_2 = 0.15$	$C_3 = 0.2$	$C_4 = 0.25$
$R_{12} = 0.15$	$R_{34} = 0.15$	$R_{23} = 1.5$	

Substituting (19) into (20), we have, for any partition function ϕ

$$\bar{V}(\phi) = \sum_{k\in\mathcal{M}} \sum_{i\in\phi^{-1}(k)} C_i T_i(0) = V.$$

This means that the aggregation does not change the invariant quantity (total heat) of the thermal system.

The "optimal" model described so far is for a given fixed partition function ϕ. The optimal partition function ϕ^* is obtained via recursive application of the spectral partitioning method.

B. Modeling error defined as KL divergence rate

We denote the full-order building thermal model as Υ and the optimal reduced-order model with m super-states as $\bar{\Upsilon}^{(m)}$.

We define the modeling error between Υ and $\bar{\Upsilon}^{(m)}$ in terms of the KL divergence rate. By following the methodology presented in Section III, we obtain the KL divergence rate as

$$\begin{aligned} R(\Upsilon \parallel \bar{\Upsilon}^{(m)}) = &\sum_{i,j\in\mathcal{N}, i\neq j} \pi_i (A_{ij}\log(A_{ij}) - A_{ij}) \\ &- \sum_{i,j\in\mathcal{N}, \phi(i)\neq\phi(j)} \pi_i A_{ij} \log\Big(\frac{\pi_j}{\sum_{k\in\psi(j)}\pi_k} \bar{A}_{\phi(i)\phi(j)}\Big) \\ &- \sum_{i,j\in\mathcal{N}, \phi(i)=\phi(j)} \pi_i (\bar{A}_{\phi(i)\phi(j)} - A_{ii} \log\frac{\pi_j}{\sum_{k\in\psi(j)}\pi_k}), \end{aligned} \tag{21}$$

where π is the stationary distribution of the full-order model Υ, A denotes the transition-rate matrix of Υ, $\phi = \phi^*$ is the optimal m-partition function, $\bar{A} = \bar{A}(\phi^*)$ denotes the super-transition-rate matrix for the optimal reduced model $\bar{\Upsilon}^{(m)}$.

V. EXAMPLE AND DISCUSSION

In this section, we apply the model reduction method developed in the previous section to two examples.

A. Model reduction of a 2-adjacent-room building

First, we consider the model reduction problem for the basic element of any building thermal model: two adjacent rooms without other thermal interactions (see Fig. 2). The thermal dynamics can be represented by an equivalent RC-network as also shown in Fig. 2, where T_1 and T_4 are the temperatures for Room 1 and Room 2 respectively, T_2 and T_3 are the internal temperatures for the connecting walls.

The building thermal model is defined in (1), where the transition rate matrix for this example is given by

$$A = \begin{bmatrix} -\frac{1}{C_1R_{12}} & \frac{1}{C_1R_{12}} & 0 & 0 \\ \frac{1}{C_2R_{21}} & -\frac{R_{21}+R_{23}}{C_2R_{21}R_{23}} & \frac{1}{C_2R_{23}} & 0 \\ 0 & \frac{1}{C_3R_{32}} & -\frac{R_{32}+R_{34}}{C_3R_{32}R_{34}} & \frac{1}{C_3R_{34}} \\ 0 & 0 & \frac{1}{C_4R_{43}} & -\frac{1}{C_4R_{43}} \end{bmatrix},$$

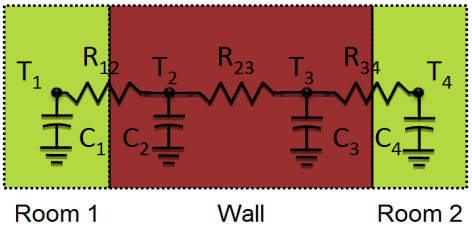


Fig. 2. The RC-network representation of the full-order thermal model for a 2-adjacent-room building without other thermal interactions.

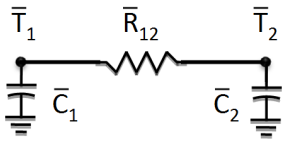


Fig. 3. The RC-network representation of the reduced-order thermal model for the 2-adjacent-room building.

where numerical values of the capacitances and resistances are given in Table I.

By choosing the sampling time $\Delta t = 0.01$, we can approximate the continuous model (1) by a discrete-time Markov chain, where the probability transition matrix is approximated by

$$P(\Delta t) \approx I + A\Delta t = \begin{bmatrix} 0.3333 & 0.6667 & 0.0000 & 0.0000 \\ 0.4444 & 0.5111 & 0.0444 & 0.0000 \\ 0.0000 & 0.0333 & 0.6333 & 0.3333 \\ 0.0000 & 0.0000 & 0.2667 & 0.7333 \end{bmatrix},$$

whose stationary distribution is given by $\pi = [0.1429, 0.2143, 0.2857, 0.3571]$. The following symmetric matrix is then obtained for solving the eigenvalue problem

$$\hat{P} = \begin{bmatrix} 0.3333 & 0.5443 & 0.0000 & 0.0000 \\ 0.5443 & 0.5111 & 0.0385 & 0.0000 \\ 0.0000 & 0.0385 & 0.6333 & 0.2981 \\ 0.0000 & 0.0000 & 0.2981 & 0.7333 \end{bmatrix},$$

whose second largest eigenvalue $\lambda_2 = 0.9605$ and the corresponding eigenvector is given by

$$u^{(2)} = [-0.5251, -0.6049, +0.3628, +0.4762].$$

The sign-structure of $u^{(2)}$ suggests the optimal bi-partition function $\phi^* = [1, 1, 2, 2]$. That is, the states $\{1, 2\}$ should be aggregated as the first group and the states $\{3, 4\}$ should be aggregated as the second group. This optimal partition is also consistent with the fact that the Markov chain (π, P) is *nearly completely decomposable*.

Using (16) with the optimal bi-partition function ϕ^*, we obtain the super-transition-rate matrix for the optimal reduced-order model

$$\bar{A} = \begin{bmatrix} -\frac{1}{\bar{C}_1 \bar{R}_{12}} & \frac{1}{\bar{C}_1 \bar{R}_{12}} \\ \frac{1}{\bar{C}_2 \bar{R}_{21}} & -\frac{1}{\bar{C}_2 \bar{R}_{21}} \end{bmatrix},$$

where numerical values of the super-capacitances and the super-resistance are summarized in Table II. The RC-network representation of the reduced-order model is shown in Fig. 3.

This example also shows that the proposed method yields a reduced model that is physically meaningful. Since the resistance (R_{12} or R_{34}) between the two capacitors (C_1 and C_2, or C_3 and C_4) is small, then the two capacitors are combined into one. Since the internal resistance R_{23} is the predominant one, it is retained in the reduced model. Finally, the reduced model is also an RC-network just as the original model was.

TABLE II

NUMERICAL VALUES FOR THE REDUCED-ORDER MODEL

$\bar{C}_1 = C_1 + C_2 = 0.25$	$\bar{C}_2 = C_3 + C_4 = 0.45$
$\bar{R}_{12} = \bar{R}_{21} = R_{23} = 1.5$	

B. Model reduction of the 4-room building

We consider the 4-room building as shown in Fig. 1 (a). There are four rooms inside the building. Rooms are connected with each other through the internal walls, and also with the outside through ceilings, floors, windows and external walls. To simplify the model, we assume virtually no air exchange between rooms occur through the doors.

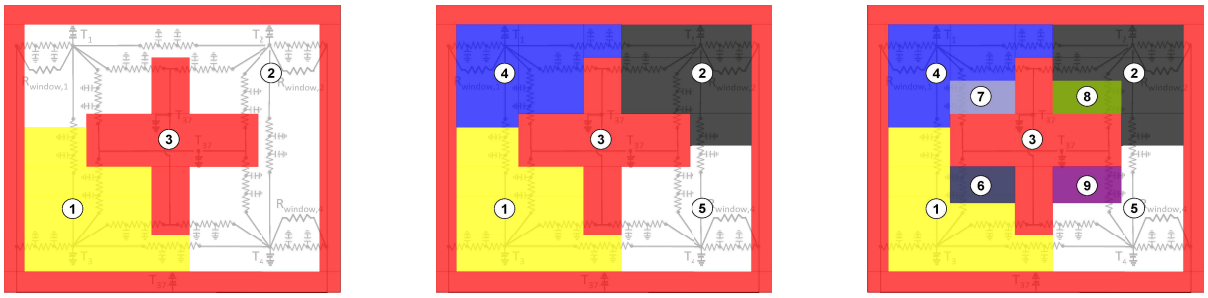
By inter-connecting the RC-network models for individual walls, floors and ceilings, we represent the entire building by a large RC-network as shown in Fig. 1 (b). There are total 37 nodes of this RC-network, associated with 37 temperatures:

- $\{T_1, T_2, T_3, T_4\}$ for the 4 room nodes;
- $\{T_5, T_6, \ldots, T_{12}\}$ for the 8 internal wall nodes;
- $\{T_{13}, T_{14}, \ldots, T_{20}\}$ for the 8 internal floor nodes;
- $\{T_{21}, T_{22}, \ldots, T_{28}\}$ for the 8 internal ceiling nodes;
- $\{T_{29}, T_{30}, \ldots, T_{36}\}$ for the 8 external wall nodes;
- $\{T_{37}\}$ for the 1 outside node.

Generally, each node i is assigned a capacitance C_i, two adjacent nodes i and j are connected with a resistance R_{ij}. The windows are modeled as single resistors since they have relatively little capacitance. The outside air has a much larger capacitance than the air inside the rooms. So, we assign a very large value to its capacitance, i.e., $C_{37} = 10^{10}$.

The transition-rate matrix A for the building thermal model is a 37×37 matrix obtained according to the building topology and associated capacitances and resistances, which are calculated for a specific wall material and insulation type as explained in Section I. By choosing the sampling time $\Delta t = 0.01$, we obtain the transition matrix $P = e^{A\Delta t}$ for the Markov chain representation of the building thermal model. Then the recursive bi-partition algorithm [9] is employed to solve the m-partition problem and obtain the optimal reduced-order model with m super-nodes. With $m = 1$, all nodes belong to the same group. The modeling error is given by $R(\Upsilon \parallel \bar{\Upsilon}^{(1)}) = 3.67 \times 10^{-4}$.

The bi-partition divides the node set into two groups: the first group contains all non-outside nodes: $\{1, 2, \ldots, 36\}$, and the second group contains only the outside node: $\{37\}$. Such an aggregation makes sense because with a large time-scale, the temperatures of nodes inside the building will



(a) (b) (c)

Fig. 4. The group information of building nodes for (a) the 3-partition, (b) the 5-partition, and (c) the 9-partition. Note that the red region (the third group) is corresponding to a single "outside" node $\{37\}$ in the graph.

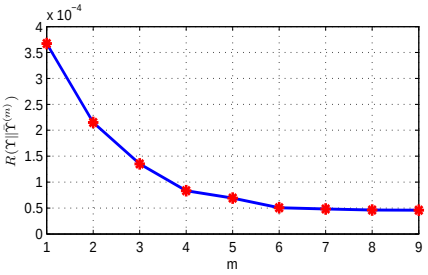


Fig. 5. Depicts the modeling error as a function of the number of partitions.

all asymptotically approach the outside temperature. In a Markov chain representation, the outside node is a nearly absorbing state and all other states are nearly transient. The Modeling error for the bi-partition is given by $R(\Upsilon \parallel \bar{\Upsilon}^{(2)}) = 2.15 \times 10^{-4}$.

The 3-partition divides the node set into three groups by further partitioning the set of nodes inside the building: the first group consists of all (wall, ceiling and floor) nodes associated with Room 3: $\{3, 6, 9, 10, 11, 15, 19, 23, 27, 31, 35\}$, the second group contains all other nodes associated with Rooms $1, 2, 4$, and the third group contains only the outside node: $\{37\}$ (see Fig. 4 (a)).

By recursively applying the bi-partition algorithm, we can partition the node set into more groups. Figure 4 (b) illustrates the 5-partition results. After the 5^{th} partition, we have the following five groups: one group containing only the outside node, and the other four groups with each group containing the nodes associated with one of the four rooms. A partition with more than 5 super-states (i.e., $m > 5$) will partition the nodes inside individual rooms. For example, Figure 4 (c) depicts the partition results for the 9-partition problem.

Fig. 5 depicts the modeling error $R(\Upsilon \parallel \bar{\Upsilon}^{(m)})$ as a function of m (the number of super-nodes) for the first nine partitions. The modeling error decreases rapidly from $m = 1$ to $m = 6$. For $m > 6$, the modeling error decreases more gradually. The modeling error plot suggests that the model with 6 super-nodes is the "most-appropriate" reduced-order model: Reducing a super-node causes the modeling error to increase by a large amount, while adding additional super-nodes leads to only a small amount of reduction for the modeling error.

VI. Conclusions

In this paper, a building thermal model is reduced via the aggregation of states. The original model is an RC-network, with a large number of coupled linear differential equations. The conservative nature of the thermal model is used to draw an analogy to regular Markov chains. The KL divergence rate serves as a metric for the modeling errors. A key advantage of the proposed technique is that reduced model retains the physical intuition of the original model: it is also an RC-network model. An additional advantage of the method is that the degree of reduction can be controlled by the user.

Acknowledgment

Kun Deng and Prashant G. Mehta's research was supported in part by NSF grants ECCS-0925534. Prabir Barooah's work was partially supported by NSF CAREER grant ECCS-0955023. Sean P. Meyn's research was supported in part by NSF grant CNS-0931416.

References

[1] P. Torcellini, M. Deru, B. Griffith, N. Long, S. Pless, and R. Judkoff, "Lessons learned from case studies of six high performance buildings," National Renewable Energy Laboratory, Tech. Rep. NREL/TP-50-37542, 2006.

[2] J. E. Seem, S. A. Klein, W. A. Beckman, and J. W. Mitchell, "Transfer functions for efficient calculation of multidimensional transient heat transfer," *Journal of Heat Transfer*, vol. 111, no. 1, pp. 5–12, 1989.

[3] J. E. Braun, K. W. Montgomery, and N. Chaturvedi, "Evaluating the performance of building thermal mass control strategies," *Heating Ventillation Air Conditioning and Refrigeration Research*, vol. 7, no. 4, pp. 403–428, 2004.

[4] X. Xu and S. Wang, "A simplified dynamic model for existing buildings using CTF and thermal network models," *International Journal of Thermal Sciences*, vol. 47, no. 9, pp. 1249–1262, 2008.

[5] M. M. Gouda, S. Danaher, and C. Underwood, "Building thermal model reduction using nonlinear constrained optimization," *Building and Environment*, vol. 37, no. 12, pp. 1255–1265, 2002.

[6] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. Philadelphia, PA: Society for Industrial and Applied Mathematics, 2005.

[7] M. Zaheer-Uddin, "Energy start-stop and fluid flow regulated control of multizone hvac systems," *Energy*, vol. 18, no. 3, pp. 289–302, 1993.

[8] M. Mossolly, K. Ghali, and N. Ghaddar, "Optimal control strategy for a multi-zone air conditioning system using a genetic algorithm," *Energy*, vol. 34, no. 1, pp. 58–66, 2009.

[9] K. Deng, Y. Sun, P. G. Mehta, and S. P. Meyn, "An information-theoretic framework to aggregate a Markov chain," in *Proceedings of American Control Conference*, St. Louis, MO, 2009, pp. 731–736.