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Error growth in position estimation from noisy relative pose measurements

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ABSTRACT

We examine how the estimation error grows with time when a mobile robot estimates its location from relative pose measurements without global position or orientation sensors. We show that, in both two-dimensional and three-dimensional space, both the bias and the variance of the position estimation error grows at most linearly with time asymptotically. Non-asymptotic bounds on the bias and variance are obtained, which provide insight into the mechanism of error growth. The bias is crucially dependent on the trajectory of the robot. Conclusions on the asymptotic growth rate of the bias continue to hold even with unbiased measurements or error-free translation measurements. Exact formulas for the bias and the variance of the position estimation error are provided for two specific two-dimensional trajectories – straight line and periodic. Experiments with a P3-DX wheeled robot and Monte Carlo simulations are provided to verify the theoretical predictions. A method to reduce the bias is proposed based on the lessons learned.

1. Introduction

Localization without GPS is a key capability for autonomous robots, since there are many situations in which GPS signals are either unavailable or only intermittently available. These include operation in urban canyons and tunnels, inside buildings, under water, and in extra-planetary exploration. In such a situation, localization with respect to an initial position is typically performed using a combination of sensors that are used to measure relative motion between two successive time instants, and then chaining them together. Inertial sensors (gyroscopes and accelerometers), vision-based sensors (cameras, LIDARs, etc.) and joint encoders (in the case of ground vehicles) are examples of sensors that can be used to obtain such measurements. Apart from robotic platforms, such localization is also of relevance to human wearable systems [1], personal navigation devices [2], and robot end-effector position estimation [3].

In this paper we examine the growth rate of the position estimation error of a robot that cannot directly measure either its global position or its global orientation. Specifically, we analyze the bias and the variance of the error. The robot is equipped with sensors that allow it to measure the relative pose (position and orientation) between its coordinate frames at two successive time instants, but not sensors that can measure its absolute pose with respect to a global coordinate frame. That is, the robot may have sensors such as wheel odometers, IMUs, and cameras, but does not have sensors such as GPS and compass. The absolute position has to be estimated from the noisy relative pose measurements.

When relative pose measurements obtained from sensors are concatenated to form an estimate of the robot's position in a global frame, errors in individual measurements accumulate. Over long time horizons, the resulting location estimates may become quite poor. Though this is well recognized, a rigorous analysis of the growth rate is lacking. Both Volpe [4] and Olson et al. [5] present experimental evidence of the error growing superlinearly with distance without a global orientation sensor. Volpe [4] provides error growth analysis of a limited scenario (two-dimensional straight line with bias-free IMU) and concludes that for that scenario, the error is quadratic in distance traveled when the time approaches 0 and linear in distance traveled when the time approaches ∞. Olson et al. [5] state that the position estimation error will grow as $O(s^{3/2})$, where s is the distance traveled. In fact, a number of papers have cited [5] in order to state that the error grows superlinearly in the absence of an absolute orientation sensor [6–10]. A parametric statistical model of the 2-norm of the position estimation error is proposed in [11], whose parameters have to be fitted from measured error. In contrast to [11], the papers [4–10] seem to describe the error in deterministic terms rather than in statistical measures such as mean and variance.

In this paper we rigorously analyze the bias and the variance of the position estimation error, and obtain asymptotic as well as non-asymptotic bounds for them. We show that the asymptotic growth rates of both the bias and the variance are upper bounded by linear functions of time. Thus, even without an absolute orientation sensor, the error growth (for both the bias and the

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variance) is at most linear. We also show that the variance growth rate is lower bounded by a linear function of time as well, if the variance of the translation measurement is sufficiently large. A method for reducing the bias in the localization error is proposed that is based on the insights obtained. One of the key insights is that the distance traveled is immaterial in determining either the bias or the variance of the localization error.

The results mentioned above, which are stated in Theorem 1 of the paper, are for the general d-dimensional case: the robot's pose is an element of $SE(d)$, $d \in \{2, 3\}$. For two special trajectories in two dimensions, namely straight line and periodic, we provide exact formulas for the bias and the variance of the position estimation error (Theorems 2 and 3). Analysis of the two-dimensional case is more tractable than the three-dimensional case since two-dimensional rotation matrices commute. We see from both the two-dimensional and three-dimensional results that the bias and variance do indeed appear to grow faster than linearly with time for small time intervals. The linear asymptotic trend is visible only when the time is sufficiently large. These results are verified numerically through Monte Carlo simulations. We also provide experimental verification of the periodic case through experiments conducted with a Pioneer P3-DX robot equipped with a vision-based sensor and a wheel odometer.

The error growth rate results established here provide useful benchmarks to several localization-related applications. A specific instance is an analysis of the benefits of various types of measurements for vehicle localization. The advantage of absolute position and/or orientation measurements, even if intermittently available, to aid dead-reckoning localization is well established experimentally. However, the utility of other types of absolute measurements in such a scenario, such as intermittent measurements of the distance to fixed beacons[1] or relative orientation between the vehicle and a fixed frame, does not seem to be have been comprehensively investigated. If an analysis is carried out to obtain growth rates of the localization error with various types of measurements, the results of this paper can be used to compare the benefit of such a localization scheme with that from dead-reckoning alone. Such a comparison is useful to determine if the benefits, if any, warrant the investment required (both hardware and software) to obtain such measurements.

Another application where our results can serve as a benchmark is collaborative localization of multiple robots, in which robot-to-robot measurements are fused with dead-reckoning measurements of individual robots to improve the localization accuracy of all the robots. Imagine the following hypothetical scenario. A particular collaborative localization algorithm is proved to have an $O(s/N)$ asymptotic growth rate of the, say, bias of the error, where s is the distance traveled and N is the number of robots. When examined compared to the prior belief that the error grows as $O(s^{3/2})$ in the single robot case, such a result would indicate that the collaborative localization algorithm is more beneficial than it actually is.

The results in the paper also have more practical implications. These contributions come from the lessons learned in performing the analysis of error growth. First, our analysis provides insight into the mechanism of error growth, particularly its bias. Specifically, we show that the expected value of the robot's estimated position always converges to a point. This occurs because the measurement of the translation during a time step, when transformed into the global frame to add to the previous position estimate, decays in magnitude geometrically with time. This geometric decay occurs due to the norm of the rotation error being less than one (Proposition 1). As a result, the growth of the bias depends crucially on the type of path the robot traverses even though the robot does not have – and does not use – information about its trajectory. The bias will be bounded or unbounded depending only on whether the robot stays within a bounded region or not. In addition, the asymptotic trends for the bias hold even if the measurements of relative translation and rotation are unbiased. In fact, they hold even if the relative translation measurements are completely error free. The bias in the translation *measurements* that arise from vision-based sensors has been a topic of research [12,13]. However, the fact that large position estimation bias may occur even when all measurements are unbiased has not been emphasized in the literature.

An important insight to be gained from the analysis is that the *distance traveled* by the robot is immaterial in determining the bias and variance of the position estimation error; but the magnitude of the displacement vector (current position with respect to the starting point) is a key determinant of the bias. It is only when the robot moves in an approximately straight path that the distance traveled and the displacement are similar. It is common in the literature to characterize the performance of a localization scheme in terms of the error expressed as a percentage of the distance traveled. Our results indicate this can be misleading, especially when the robot stays in a bounded region.

A second practical contribution is the proposed method for reducing the bias in the localization error. The challenge in bias reduction is that the bias in the absolute translation estimate involves unknown quantities such as the true translation. Otherwise one could simply compute this bias and subtract it from the translation estimate before combining with the previous position estimate. Instead, the proposed method for bias reduction modifies the relative rotation estimates to slow down the geometric reduction of the translation estimate that occurs otherwise. It is observed in simulations that the method is able to reduce the bias to close to zero. Preliminary results are included in the paper, with a more thorough investigation of the method planned for the future.

The rest of the paper is organized as follows. Section 1.1 discusses some related work. Section 2 precisely formulates the problem under study, and Section 3 states the main results. Most of the proofs are in the Appendix at the end of the paper. Simulation verification is presented in Section 4 and experimental verification is presented in Section 5. A method to reduce the bias in position error is given in Section 6. The paper ends with a discussion of the results in Section 7.

1.1. Related work

The papers by Smith and Cheeseman [3], Su and Lee [14], and Wang and Chirikjian [15] derived recursive expressions for the covariance of the pose estimation error by assuming that the errors are small, so that a first-order approximation of the BCH (Baker–Campbell–Hausdorff) formula is valid. Recently, Wang and Chirikjian [16] developed a recursive formula for the covariance of the pose estimation error that retains the second-order terms in the BCH formula. The paper [17] examines the dead-reckoning error's probability density function for non-holonomic robots in two dimensions. The works that come close to ours in spirit are [11,4]. In [11], a parametric statistical model of the 2-norm of the position estimation error is proposed. Some of the parameters have to be fitted from measured error. Some analysis of the error growth due to IMU integration is provided in [4]. It is shown that for the specific scenario examined (two-dimensional motion in a straight line without change in orientation, bias-free IMU measurement, etc.), the position error is proportional to distance traveled for large values of time. However, the works mentioned above do not analyze the rate at which the error's mean and variance grow with time.

[1] We exclude simultaneous distance measurement to three or more beacons from this discussion, since that is equivalent to an absolute position measurement.

A related body of literature deals with the problem of developing state estimation techniques for systems whose states, as well as the noisy measurements, are in $SO(3)$ or $SE(3)$ (see [18,19] and references therein). The problem of position estimation of a mobile robot with noisy relative pose measurements between successive frames – one that is central to this paper – falls into this category. However, our aim is not to develop an estimation technique, but to examine the growth of error in the position estimate when successive noisy relative pose measurements are chained together to obtain a global pose estimate.

2. Problem statement

We measure time with a discrete index $k = 0, 1, \ldots$. Sensors used for relative localization of autonomous vehicles yield an estimate of the position and orientation of the vehicle at time k relative to that in the previous time instant, $k-1$. That is, they produce an estimate of the *relative pose* between frames attached to the robot at two successive time instants. Let $\mathbf{R}^k_{k+1}$ be the rotation between the local frames attached to the robot's body at time k and $k+1$. That is, if $\mathbf{u}^k$ is a vector expressed in the vehicle's frame at time k and $\mathbf{u}^{k-1}$ is the same vector expressed in the vehicle's frame at time $k-1$, then $\mathbf{u}^{k-1} = \mathbf{R}^{k-1}_k\ \mathbf{u}^k$. This notation is adopted from [20]. We will refer to the frame that is attached to the vehicle at time k as "frame k". Similarly, let $\mathbf{t}^k_{i,j}$ be the relative translation from frame i to frame j, expressed in frame k. The rotation $\mathbf{R}^{k-1}_k \in SO(d)$ is usually expressed as a $d \times d$ matrix for $d \in \{2, 3\}$, while $\mathbf{t}^k_{i,j}$ is a vector in $\mathbb{R}^d$. Without loss of generality, the coordinate frame that is attached to the robot's body at the initial time $k = 0$ is used as the global coordinate frame. We denote the rotation from frame k to the global coordinate frame (frame 0) by $\mathbf{R}^0_k$. Similarly, the translation from frame $k-1$ to frame k expressed in the global coordinate frame is denoted by $\mathbf{t}^0_{k-1,k}$. The position of the robot at time n is the vector $\mathbf{t}^0_{0,n}$.

With relative pose sensors such as cameras, inertial sensors, and wheel odometers, the measurements available at time k are estimates of the relative translation from frame $k-1$ to frame k expressed in frame k, i.e., of $\mathbf{t}^k_{k-1,k}$, and the rotation between the frames $k-1$ and k, i.e., of $\mathbf{R}^{k-1}_k$. The translation from $k-1$ to k, for $k \geq 1$, expressed in the global coordinate frame, is

$$\mathbf{t}^0_{k-1,k} = \mathbf{R}^0_k\,\mathbf{t}^k_{k-1,k}, \quad \text{where } \mathbf{R}^0_k = \mathbf{R}^0_1\,\mathbf{R}^1_2 \cdots \mathbf{R}^{k-1}_k.$$

An example of a robot's path along with its corresponding relative pose measurements can be seen in Fig. 1. Estimates are denoted by hats on top of the corresponding symbols, and errors by tildes, so that $\hat{\mathbf{R}}^{k-1}_k$ and $\hat{\mathbf{t}}^k_{k-1,k}$ are the noisy estimates of $\mathbf{R}^{k-1}_k$ and $\mathbf{t}^k_{k-1,k}$, and the corresponding errors $\tilde{\mathbf{R}}^{k-1}_k$ and $\tilde{\mathbf{t}}^k_{k-1,k}$ are defined as

$$\begin{aligned}\tilde{\mathbf{R}}^{k-1}_k &:= (\mathbf{R}^{k-1}_k)^{-1}\hat{\mathbf{R}}^{k-1}_k,\\ \tilde{\mathbf{t}}^k_{k-1,k} &:= \hat{\mathbf{t}}^k_{k-1,k} - \mathbf{t}^k_{k-1,k}.\end{aligned} \tag{1}$$

The absolute position of the robot at time k is determined by adding the relative position measurements, after expressing them all in the global coordinate frame. The measurement of the translation from frame $k-1$ to frame k expressed in the global coordinate frame, which is denoted by $\hat{\mathbf{t}}^0_{k-1,k}$, is

$$\hat{\mathbf{t}}^0_{k-1,k} := \hat{\mathbf{R}}^0_k\,\hat{\mathbf{t}}^k_{k-1,k}, \tag{2}$$

where $\hat{\mathbf{R}}^0_k$ is an estimate of $\mathbf{R}^0_k$, which is computed from the relative rotation estimates as

$$\hat{\mathbf{R}}^0_k = \prod_{i=1}^{k} \hat{\mathbf{R}}^{i-1}_i. \tag{3}$$

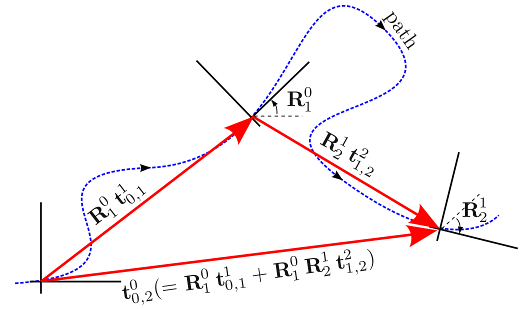


Fig. 1. A figure to explain the notation: a robot's path (shown in dashed blue line) in two dimensions and associated relative poses between time instants. $\mathbf{t}^0_{k-1,k}$ is the translation between the frames $k-1$ and k, expressed in the global frame 0, and $\mathbf{t}^k_{k-1,k}$ is the same vector expressed in the local frame k. The matrix $\mathbf{R}^0_k$ is the rotation between frame 0 and frame k, so that $\mathbf{R}^0_k\mathbf{t}^k_{k-1,k}$ is the translation from $k-1$ to k expressed in the global frame 0. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Finally, the estimate of the position at time n in the global coordinate frame 0 is obtained by adding the relative translation estimates after transforming them to frame 0:

$$\hat{\mathbf{t}}^0_{0,n} := \sum_{k=1}^{n} \hat{\mathbf{t}}^0_{k-1,k}. \tag{4}$$

The error between the estimated position and the true position at time n is

$$\mathbf{e}(n) := \mathbf{t}^0_{0,n} - \hat{\mathbf{t}}^0_{0,n}. \tag{5}$$

The goal of this paper is to study how the mean and covariance of the position estimation error $\mathbf{e}(n)$ *scales with the time index* n. If the robot's speed is upper and lower bounded by two constants, then the asymptotic trends with time are equivalent to those with distance traveled.

The straightforward dead-reckoning formula (4) may not be used in practice. Typically a filtering-based algorithm is used to fuse relative pose measurements with the predictions of a model of the robot's motion. There are many variations possible in terms of assumed model, states and input measurements; see [21] for a comparison among some of them. This renders examining the mechanism of error propagation and establishing growth rates with such algorithms intractable. Therefore we adopt the simple dead-reckoning model that still captures the essential features of localization from relative pose measurements. We wish to emphasize that the estimation error resulting from the estimation method described above will have the same asymptotic trend as that of a filtering technique that uses a kinematic model of the robot motion. The reason is that a kinematic model essentially produces an independent noisy measurement of the relative pose. Thus, our investigations are useful in analyzing asymptotic performance of a wider class of estimation techniques. One situation where our model is *not* appropriate is when vision-based loop closure is used to augment localization [22]. We focus on situations where loop closure is not applicable, e.g., an unmanned aerial vehicle flying in an expansive environment so that it may not come back to its earlier positions.

To state the assumptions on measurement error statistics, we establish a few conventions. A rotation matrix $R \in SO(3)$, where the special orthogonal group $SO(3)$ is the set of 3×3 real orthogonal matrices with unit determinant, can be represented by the exponential map: $R = e^{\omega^s}$, where ω^s is the 3×3 skew-symmetric matrix corresponding to the vector $\omega \in \mathbb{R}^3$ [23, Chapter 2]. A matrix in $SO(2)$ is uniquely specified by an angle $\theta \in [-\pi, \ \pi)$. A random rotation matrix $\mathbf{R} \in SO(3)$ (respectively, $SO(2)$) can therefore be specified by a random vector $\boldsymbol{\omega} \in \mathbb{R}^3$

(respectively, a scalar random variable θ). We say that two random rotation matrices $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$ are independent if their corresponding $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are independent random vectors. For $SO(2)$, independence of rotations is defined as the independence of the scalar random variables θ_1, θ_2 that uniquely determine the rotations. If $\mathbf{R}_1$ and $\mathbf{R}_2$ are independent, every entry of the matrix $\mathbf{R}_1$ is independent of every entry of $\mathbf{R}_2$. Similarly, we say that a rotation $\mathbf{R}_1 \in SO(3)$ (respectively, $SO(2)$) and a random vector $\mathbf{t} \in \mathbb{R}^3$ (respectively, $\mathbb{R}^2$) are independent if $\boldsymbol{\omega}_1$ (respectively, θ) and $\mathbf{t}$ are independent. In this case, too, every entry of $\mathbf{t}$ is independent of every entry of $\mathbf{R}$.

In this paper, we use $\mathrm{E}[\mathbf{R}]$ (for a random rotation matrix $\mathbf{R}$) to denote the matrix whose i,j-th entry is $\mathrm{E}[(\mathbf{R})_{i,j}]$, i.e., the expected value of the i,j-th entry of $\mathbf{R}$. As a result of this convention, if $\mathbf{R}_1 \in SO(d)$ is independent of $\mathbf{R}_2 \in SO(d)$ and of $\mathbf{t} \in \mathbb{R}^3$, then $\mathrm{E}[\mathbf{R}_1\mathbf{R}_2] = \mathrm{E}[\mathbf{R}_1]\mathrm{E}[\mathbf{R}_2]$ and $\mathrm{E}[\mathbf{R}_1\mathbf{t}] = \mathrm{E}[\mathbf{R}_1]\mathrm{E}[\mathbf{t}]$.

In what follows, $\mathrm{Tr}[\cdot]$ stands for the trace of a matrix, and $\|\cdot\|_q$ denotes the (induced) q-norm of a (matrix) vector. When the subscript is omitted, it denotes the (induced) 2-norm.

We state the following assumptions for use in the rest of the paper.

Assumption 1. 1. The robot's speed is uniformly bounded. More specifically, there exists a constant $\tau > 0$ such that $\|\mathbf{t}^k_{k-1,k}\| \leq \tau$.

2. The translation measurement errors $\tilde{\mathbf{t}}^k_{k-1,k}$ form a sequence of independent random vectors, with mean $\mathbf{b}_k := \mathrm{E}[\tilde{\mathbf{t}}^k_{k-1,k}]$ and covariance $\mathbf{P}_k := \mathrm{Cov}(\tilde{\mathbf{t}}^k_{k-1,k}, \tilde{\mathbf{t}}^k_{k-1,k})$ that are uniformly bounded. That is, there exist scalar constants $b, \underline{p}, \overline{p}$ such that $0 \leq \|\mathbf{b}_k\| \leq b$ and $0 \leq \underline{p} \leq \mathrm{Tr}[\mathbf{P}_k] \leq \overline{p} < \infty$ for all k.
3. The rotation measurement errors $\tilde{\mathbf{R}}^k_{k+1}$ form a sequence of independent random matrices. The rotation and translation measurement errors $\tilde{\mathbf{R}}^{j-1}_j$ and $\tilde{\mathbf{t}}^k_{k-1,k}$ are mutually independent if $j \neq k$, and possibly dependent when $j = k$, with $\mathrm{E}[\tilde{\mathbf{R}}^{k-1}_k \; \tilde{\mathbf{t}}^k_{k-1,k}] =: \boldsymbol{\rho}_k \in \mathbb{R}^d$. There exists a scalar ρ such that $\|\boldsymbol{\rho}_k\| \leq \rho$ for all k.
4. The relative translation measurement errors $\{\tilde{\mathbf{t}}^k_{k-1,k}\}_{k=1}^{\infty}$ are uniformly absolutely integrable, i.e., there exists a scalar β so that $\beta_k \leq \beta < \infty$ for all k, where $\beta_k := \mathrm{E}\|\tilde{\mathbf{t}}^k_{k-1,k}\|$.
5. The rotation measurement errors $\tilde{\mathbf{R}}^k_{k+1}$ are identically distributed, so that each $\tilde{\mathbf{R}}^k_{k+1}$ has the same distribution as that of some matrix $\tilde{\mathbf{R}} \in SO(d)$, $d \in \{2, 3\}$. Moreover, $\tilde{\mathbf{R}}$ is not degenerate, i.e., its pdf (probability distribution function) is not concentrated on a set of measure zero.

Apart from the assumptions on independence of measurement errors, the other assumptions, namely those on the existence of the parameters $\tau, b, \underline{p}, \overline{p}, \rho, \beta$, are trivially satisfied in any practical scenario. Finiteness of the displacement τ and the parameter b is easy to see; the parameters $\underline{p}$ and $\overline{p}$ are simply the lower and upper bounds on the eigenvalues of $\mathbf{P}_k$. The d-dimensional vector $\boldsymbol{\rho}_k$ is a measure of the correlation between the translation and rotation measurements, and the parameter ρ is an upper bound on the magnitude of the correlation. We allow the relative translation and rotation measurement errors at a particular time instant to be statistically dependent, since this may happen if there is overlap between the sensor suite used to obtain these two measurements. The parameter β is akin to an upper bound on the sum of bias and variance of the translation measurement error. To see this, consider not $\mathrm{E}[\|\tilde{\mathbf{t}}^k_{k-1,k}\|]$, but $\mathrm{E}[\|\tilde{\mathbf{t}}^k_{k-1,k}\|^2]$, which is the trace of the second moment of translation measurement error $\tilde{\mathbf{t}}^k_{k-1,k}$. Since the second moment is the sum of covariance and square of the first moment, an upper bound on $\mathrm{E}[\|\tilde{\mathbf{t}}^k_{k-1,k}\|^2]$ is also an upper bound on sum of mean and variance (more precisely, on $\|\mathbf{b}_k\|^2 + \mathrm{Tr}[\mathbf{P}_k]$) of the translation measurement error.

The following technical result is crucial for the main results of this paper and will be required for the subsequent discussions. We therefore state it here; the proof is provided in the Appendix.

Proposition 1. *Let $\mathbf{R}$ be a random rotation matrix with distribution defined over $SO(d)$, $d \geq 2$, and let $\mathrm{E}[\mathbf{R}]$ be the $d \times d$ matrix whose i,j-th entry is the expected value of the i,j-th entry of $\mathbf{R}$. We have $\|\mathrm{E}[\mathbf{R}]\| \leq 1$, and the inequality is strict if the distribution of $\mathbf{R}$ is not degenerate.*[2] □

3. Main results

3.1. General trajectories

Before stating the result, we recall the asymptotic O, Ω, Θ notation. For two scalar-valued functions $f(n), g(n)$ taking non-negative integer arguments, the notation $f(n) = O(g(n))$ means that there exists a positive integer n_1 and a positive constant c_1 such that $f(n) \leq c_1 g(n)$ for all $n \geq n_1$. The notation $f(n) = \Omega(g(n))$ means there exists a positive integer n_2 and a positive constant c_2 such that $f(n) \geq c_2 g(n)$ for all $n \geq n_2$. The notation $f(n) = \Theta(g(n))$ means that both $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$ hold.

Recall that $\tilde{\mathbf{R}}$ is a rotation matrix that has the same distribution as all the rotation errors $\tilde{\mathbf{R}}^k_{k+1}$, $k = 1, \ldots$. It follows from Proposition 1 that, under Assumption 1,

$$1 > \gamma := \|\mathrm{E}[\tilde{\mathbf{R}}]\|. \tag{6}$$

Theorem 1. *Consider a robot moving in a two-dimensional or three-dimensional Euclidean space that performs position estimation from relative pose measurements as described in Section 2. Under Assumption 1, the following statements hold, where $\tau, \beta, b, \rho, \underline{p}, \overline{p}$ are parameters defined in Assumption 1 and γ is defined in (6).*

1. *The bias in the position estimation error satisfies $\|\mathrm{E}[\mathbf{e}(n)]\| = O(n)$. In particular,*

$$\max\left\{0, \|\mathbf{t}^0_{0,n}\| - \frac{1-\gamma^n}{1-\gamma}(\gamma\tau + \beta)\right\} \leq \|\mathrm{E}[\mathbf{e}(n)]\| \leq \|\mathbf{t}^0_{0,n}\| + \frac{1-\gamma^n}{1-\gamma}(\gamma\tau + \beta). \tag{7}$$

2. *The position error covariance satisfies $\mathrm{Tr}[\mathrm{Cov}(\mathbf{e}(n), \mathbf{e}(n))] = O(n)$, with upper bound given by*

$$\mathrm{Tr}[\mathrm{Cov}(\mathbf{e}(n), \mathbf{e}(n))] \leq \overline{\alpha_0}\left(\frac{1+\gamma-2\gamma^n}{1-\gamma}\right)n, \tag{8}$$

where

$$\overline{\alpha_0} = \max\left\{(\tau^2 + 2\tau b + \overline{p} + b^2), \left(\tau + \frac{\rho}{\gamma}\right)(\tau + b)\right\}. \tag{9}$$

If, furthermore,

$$\underline{p} \geq 2b\tau + \tau^2 + 2\frac{(\tau + \rho/\gamma)(\tau + b)}{1-\gamma}, \tag{10}$$

then

$$\mathrm{Tr}[\mathrm{Cov}(\mathbf{e}(n), \mathbf{e}(n))] = \Theta(n). \quad \square$$

Before discussing the implications of the theorem, we present a result in the form of a lemma that is useful in that discussion, as well as in the proof of the theorem. The proof of the lemma is provided in the Appendix.

[2] Recall that we say that the distribution of $\mathbf{R}$ is degenerate if its pdf is 0 everywhere except possibly in a set of measure 0.

Lemma 1. *Under Assumption 1, the first and second moments of the position estimate satisfy*

$$\|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\| \le \frac{1-\gamma^n}{1-\gamma}(\gamma\tau+\beta) = O(1),$$

$$\mathrm{E}[\|\hat{\mathbf{t}}^0_{0,n}\|^2] \le \overline{\alpha_0}\frac{1+\gamma-2\gamma^n}{1-\gamma}\, n = O(n),$$

where $\overline{\alpha_0}$ is defined in (9). Moreover, if condition (10) is satisfied, then we have $\mathrm{E}[\|\hat{\mathbf{t}}^0_{0,n}\|^2] = \Theta(n)$. □

3.2. Discussion of Theorem 1 and its proof

Theorem 1, in particular the upper bound in (7), shows that, if the robot's motion is confined to a bounded region, then the bias in the position estimation error stays uniformly bounded by a constant: $\|\mathrm{E}[\mathbf{e}(n)]\| = O(1)$. If the robot moves with a constant speed and with a constant (absolute) orientation, then its position grows linearly with time. In this case the theorem tells us that the bias grows linearly with time: $\|\mathrm{E}[\mathbf{e}(n)]\| = \Theta(n)$, since now both the upper and lower bounds are asymptotically linear in time. This implies that the asymptotic trend of the bias is crucially dependent on the robot's displacement; the distance traveled is less important.

This dependency of the bias on the robot's trajectory comes from the fact that the estimated position is always bounded in mean, even if the robot is moving out to infinity, which follows from Lemma 1. To obtain an intuitive understanding of Lemma 1, we first note that the estimated position is simply the sum of the translations after transforming them to the common global coordinate frame 0; see (4). Taking the expectation on both sides of (4), we obtain

$$\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}] = \mathrm{E}[\hat{\mathbf{t}}^0_{0,1}] + \mathrm{E}[\hat{\mathbf{t}}^0_{1,2}] + \cdots + \mathrm{E}[\hat{\mathbf{t}}^0_{n-1,n}]. \tag{11}$$

The k-th term in the sum above, for the special case that rotation and translation measurements are independent, is

$$\mathrm{E}[\hat{\mathbf{t}}^0_{k-1,k}] = \mathrm{E}\left[\left(\prod_0^{k-1}\hat{\mathbf{R}}^i_{i+1}\right)\hat{\mathbf{t}}^k_{k-1,k}\right] = \left(\prod_0^{k-1}\mathrm{E}[\hat{\mathbf{R}}^i_{i+1}]\right)\mathrm{E}[\hat{\mathbf{t}}^k_{k-1,k}]$$
$$= \left(\prod_0^{k-1}\left(\mathbf{R}^i_{i+1}\mathrm{E}[\tilde{\mathbf{R}}]\right)\right)\mathrm{E}[\hat{\mathbf{t}}^k_{k-1,k}].$$

The magnitude of this term is of order γ^k, since it involves k products of $\mathrm{E}[\tilde{\mathbf{R}}]$, each of which has a norm equal to γ. Since $\gamma < 1$ (see (6)), the sum (11) is bounded for all n. The expected value of the position estimate therefore converges to a point. The bias in position estimation error, which is the mean of the difference between the estimated and true positions, is therefore dominated by the true position if the magnitude of the position vector is large. *In particular, we emphasize that the distance traversed is immaterial.*

For further discussion on the bias and variance of the position estimation error, some discussion of the statistics of the measurement errors is in order. Quantifying statistical measures of the translation measurements is straightforward. The vector $\mathbf{b}_k$, which is the mean of the translation measurement error, is called its bias, and Tr $[\mathbf{P}_k]$ is called its variance.

In contrast, quantifying rotation measurement error statistics is trickier. According to the convention used in this paper, in general $\mathrm{E}[\mathbf{R}] \notin SO(d)$ even if $\mathbf{R} \in SO(d)$. It is important that the notation $\mathrm{E}[\mathbf{R}]$ is not to be understood as the expectation of the random variable $\mathbf{R}$ with a distribution defined over $SO(d)$, which we denote by $\mu_{\mathbf{R}}$, so that $\mu_{\mathbf{R}} \in SO(d)$. We call $\mu_{\mathbf{R}}$ the "Lie-group mean" of $\mathbf{R}$. We call an estimate $\hat{\mathbf{R}}$ of a true rotation $\mathbf{R}$ *unbiased* if $\mu_{\hat{\mathbf{R}}} = \mathbf{R}$. A result of the adopted convention is that, for an unbiased estimator $\hat{\mathbf{R}}$ of $\mathbf{R}$, in general $\mathrm{E}[\hat{\mathbf{R}}] \neq \mathbf{R}$. The reason the quantity $\mathrm{E}[\mathbf{R}]$ is more useful for this paper than $\mu_{\mathbf{R}}$ is that, when $\mathbf{R}$ and $\mathbf{t}$ are independent, $\mathrm{E}[\mathbf{Rt}] = \mathrm{E}[\mathbf{R}]\mathrm{E}[\mathbf{t}]$, but in general $\mathrm{E}[\mathbf{Rt}] \neq \mu_{\mathbf{R}}\mathrm{E}[\mathbf{t}]$.

The bias in translation measurements obtained from vision-based sensors has been the subject of research [13,12]. The bias in rotation measurement, on the other hand, seem to have drawn limited attention. In [12], the error in three-dimensional rotation is described in terms of the corresponding Euler angles, and the bias in rotation is also defined in terms of the bias in the Euler angles. An alternate definition of three-dimensional rotation error in terms of a 3-vector (involving angle and axis of rotation) is used in [24], but the question of its bias is not discussed.

Notice that the bounds (7) on the bias do not depend on the error in the translation measurements. The conclusions drawn above remain the same even if the rotation and translation measurements are unbiased, i.e., $\mu_{\tilde{\mathbf{R}}} = I$, $\mathbf{b}_k = 0$, and, in fact, *even if the translation measurements are completely error free*, $\tilde{\mathbf{t}}^k_{k-1,k} = 0$.

The discussion above can be summarized into the following conclusions about the bias.

(i) For large time index n, the main contributions to the bias in the position estimate are the displacement of the robot and the errors in the relative rotation measurements.
(ii) The distance traveled by the robot is immaterial in determining either the bias or the variance.
(iii) The asymptotic scaling of the bias does not change even when the translation and rotation measurements are unbiased, and in fact even if translation measurements are completely error free.

The conclusion about rotation errors determining the position estimation error seems to be well known, and is hardly surprising. However, conclusions (ii) and (iii) do not seem to be recognized in the literature. In fact, earlier work on this topic tends to focus on the distance traveled; see [4,5].

The variance growth rate does not seem to be sensitive to the trajectory of the robot. Furthermore, unlike the bias, the variance can grow without bound when the robot's trajectory is confined to a bounded region. We will see evidence of this later in simulations and experiments reported in Sections 4 and 5. We believe that the sufficient condition (10) is conservative, and is an artifact of our proof technique. Condition (10) is usually not satisfied in practice since it requires a very large translation measurement error. Yet the position estimation error variance seems to be $\Theta(n)$ in simulations and experiments reported in Section 5.2.

The results of the theorem are in contrast to the prevalent belief in the literature that the error growth is superlinear in time if absolute orientation measurements are not available: this was first stated in [5], and then cited by [6–10]. The theorem shows that, even without absolute orientation sensors, the localization error – or more precisely its bias and variance – grows at most linearly with time. We believe that the belief about superlinear growth came about from the fact that experiments/simulations were not conducted for long enough to draw reasonable conclusions about asymptotic trends. Though the root cause is the geometric decay due to γ, since γ is usually quite close to 1, there is an initial period where the error grows sharply until the geometric decay kicks in and the linear trend becomes obvious. More insight into this phenomenon will be obtained later in Section 3.3, which discusses two-dimensional trajectories (see in particular Theorem 2). We note that our results are consistent with the experimental results presented in [5]. The formulas we provide for the error in the straight-line case in Theorem 2 does show superlinear-like growth for intermediate values of time. As mentioned earlier, linear growth becomes clear only for large values of time.

The element-wise definition of $\mathrm{E}[\tilde{\mathbf{R}}]$ lets us get away without having to define or characterize the "variance/second moment of the rotation measurement error $\tilde{\mathbf{R}}$", even in establishing bounds on the variance of the position estimation error. Later, in Theorems 2 and 3, we provide exact formulas for the variance of the position estimation error in special two-dimensional cases. Even there we do not explicitly define a variance of the rotation (angle) measurement. The key quantity is still the element-wise mean of the rotation matrix. It seems that as long as the quantity $\mathrm{E}[\tilde{\mathbf{R}}]$ can be quantified we do not need to deal with the difficult question of characterizing the variance of random variables defined over $SO(3)$. It should be noted, however, that the pdf of the rotation measurement error, which involves higher moments, does determine $\mathrm{E}[\tilde{\mathbf{R}}]$.

The proof of Theorem 1, presented next, follows from Lemma 1 in a straightforward manner.

Proof of Theorem 1. It follows from (5), by applying the triangle inequality, that

$$\|\mathrm{E}[\mathbf{e}(n)]\| \leq \|\mathbf{t}^0_{0,n}\| + \|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\| \tag{12}$$

$$\|\mathrm{E}[\mathbf{e}(n)]\| \geq \max\left\{0, \|\mathbf{t}^0_{0,n}\| - \|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\|\right\}. \tag{13}$$

From Lemma 1, we have that $\|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\|$ is upper bounded, and so the first statement follows immediately from (12) and (13).

To prove the second statement, note that

$$\begin{aligned}\mathrm{Tr}\,[\mathrm{Cov}(\mathbf{e}(n), \mathbf{e}(n))] &= \mathrm{Tr}\left[\mathrm{Cov}(\hat{\mathbf{t}}^0_{0,n}, \hat{\mathbf{t}}^0_{0,n})\right] \\ &= \mathrm{Tr}\left[\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}(\hat{\mathbf{t}}^0_{0,n})^T] - \mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]^T\right] \\ &= \mathrm{E}[(\hat{\mathbf{t}}^0_{0,n})^T\hat{\mathbf{t}}^0_{0,n}] - \|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\|^2 \\ &\leq \mathrm{E}[(\hat{\mathbf{t}}^0_{0,n})^T\hat{\mathbf{t}}^0_{0,n}].\end{aligned}$$

Since we have that $\|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\| = O(1)$, the second statement follows from Lemma 1. □

3.3. *Special two-dimensional trajectories*

In this section, we provide exact formulas for the bias and variance for the special case when the motion of the robot is confined to a two-dimensional plane and its trajectory is limited to a certain type(s). In the two-dimensional scenario $\hat{\mathbf{t}}^i_{j,k}, \mathbf{t}^i_{j,k} \in \mathbb{R}^2$ and $\mathbf{R}^i_j, \hat{\mathbf{R}}^i_j \in SO(2)$ for every i, j, k. The x-axis and the y-axis of a Cartesian coordinate frame that lies on this plane and is attached to the robot's body at the initial time $k = 0$ are used as the global coordinate frame. In the two-dimensional scenario, the robot's orientation at time n can be uniquely described by an angle $\theta_{0,n} \in [-\pi, \pi)$, which describes the rotation of its local frame about the z-axis of the global frame. The relative rotation between frames $k-1$ and k is uniquely determined by the angle by which the frame $k-1$ has to be rotated in the counterclockwise direction to reach frame k, which we denote by $\theta_{k-1,k}$. Fig. 1 shows an example. A noisy measurement of the relative rotation, denoted by $\hat{\theta}_{k-1,k}$, is assumed available at time k. The error in the relative rotation measurement is

$$\tilde{\theta}_{k-1,k} := \hat{\theta}_{k-1,k} - \theta_{k-1,k}. \tag{14}$$

For future use, we define $f_R : [-\pi, \pi) \to SO(2)$ as

$$f_R(\alpha) := \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}.$$

The matrix $\mathbf{R}^{k-1}_k$ that describes the relative rotation between frames $k-1$ and k is therefore given by $\mathbf{R}^{k-1}_k = f_R(\theta_{k-1,k})$. It can be shown from definition (1) that

$$\tilde{\mathbf{R}}^{k-1}_k = f_R(\tilde{\theta}_{k-1,k}). \tag{15}$$

The estimate of the rotation $\mathbf{R}^{k-1}_k$ therefore is $\hat{\mathbf{R}}^{k-1}_k = f_R(\hat{\theta}_{k-1,k})$.

The first result, which is stated below, is on the position estimation error growth rate when the robot moves in a straight line with constant velocity and orientation. The proof of the theorem is in the Appendix.

Theorem 2. *Consider a robot that moves on a two-dimensional plane in a straight line with a constant orientation. Formally, for all k,* $\theta_{k-1,k} = 0$ *and* $\mathbf{t}^k_{k-1,k} = \mathbf{r} \in \mathbb{R}^2$*, for some vector* $\mathbf{r}$*. In addition to Assumption 1, assume that the relative orientation error* $\tilde{\theta}$ *has a pdf that is symmetric around its mean* $\mathrm{E}[\tilde{\theta}]$*, the translation measurement errors* $\tilde{\mathbf{t}}^k_{k-1,k}$*,* $k = 1, \ldots$ *are wide sense stationary with* $\mathbf{b}_k = \mathbf{b}, \mathbf{P}_k = \mathbf{P}$*, and* $\boldsymbol{\rho}_k = \boldsymbol{\rho}$ *for all k. In that case, we have*

$$\begin{aligned}&\mathrm{E}[\mathbf{e}(n)] = n\,\mathbf{r} - \left(I - c\underline{\mathbf{R}}\right)^{-1}\left(I - (c\underline{\mathbf{R}})^n\right)\left(c\underline{\mathbf{R}}\mathbf{r} + \boldsymbol{\rho}\right), \\ &\mathrm{Tr}\,[\mathrm{Cov}(\mathbf{e}(n), \mathbf{e}(n))] = \psi n + \omega(n),\end{aligned} \tag{16}$$

where

$$c := \mathrm{E}\left[\cos\left(\tilde{\theta} - \mathrm{E}[\tilde{\theta}]\right)\right], \qquad \underline{\mathbf{R}} := f_R(\mathrm{E}[\tilde{\theta}]), \tag{17}$$

and the scalars $\psi, \omega(n)$ *are given by*

$$\begin{aligned}\psi \;&=\; 2c\mathbf{r}^T\left(I - c\underline{\mathbf{R}}\right)^{-1}\underline{\mathbf{R}}\mathbf{r} \\ &\quad + \mathrm{Tr}\left[\mathbf{P} + \mathbf{b}\mathbf{b}^T\right] + (2\mathbf{b}^T + \mathbf{r}^T)(I - c\underline{\mathbf{R}})^{-1}\boldsymbol{\rho} \\ \omega(n) \;&=\; \mathbf{r}^T(I - c\underline{\mathbf{R}})^{-2}\left(I - 4c\underline{\mathbf{R}} + 2(c\underline{\mathbf{R}})^2 + 2(c\underline{\mathbf{R}})^{n+1}\right)\mathbf{r} \\ &\quad - 2\mathbf{b}^T(I - c\underline{\mathbf{R}})^{-2}\left(I - (c\underline{\mathbf{R}})^n\right)\boldsymbol{\rho} + \mathbf{b}^T(I - c\underline{\mathbf{R}})^{-1} \\ &\quad \times \left[I - (c\underline{\mathbf{R}})^n\right]\mathbf{r} - \mathbf{r}^T(I - c\underline{\mathbf{R}})^{-2}\left[I - (c\underline{\mathbf{R}})^n\right]\boldsymbol{\rho} \\ &\quad - \left\|\left[(I - c\underline{\mathbf{R}})^{-1}(I - (c\underline{\mathbf{R}})^n)(c\underline{\mathbf{R}}\mathbf{r} + \boldsymbol{\rho})\right]\right\|_2^2.\end{aligned} \tag{18}$$

□

Since the random variable $\tilde{\theta}$ is not degenerate by Assumption 1, we have that $|c| < 1$. The spectral radius of $c\underline{\mathbf{R}}$ is strictly lower than unity since $|c| < 1$ and $\underline{\mathbf{R}} \in SO(2)$. Hence $I - c\underline{\mathbf{R}}$ is invertible and $\psi, \omega(n)$ in (18) are well defined.

An immediate corollary of Theorem 2 is that, for straight-line motion, both the bias and the variance of the position estimation error grow asymptotically linearly with time. This follows from the expressions for the bias and the variance upon using the fact that $c < 1$. However, due to the presence of the c^n terms, the growth looks superlinear for intermediate values of the time index n. Simulations described in Section 4.2 verify this statement; see in particular Figs. 4 and 5. The linear trend becomes visible only when large values of the time index n are considered. This may be one of the reasons that the error is believed to grow superlinearly with time in the literature.

The next case is a periodic trajectory in two dimensions. We say that the robot moves in a *periodic trajectory with period p* if the absolute orientation and position of the robot satisfy the following conditions: $\theta_{0,k} = \theta_{0,k+p}$ and $\mathbf{t}^0_{0,k} = \mathbf{t}^0_{0,k+p}$ for all k. The shape of the (closed) path along which the robot moves can be arbitrary. In the statement of the theorem, η denotes the number of periods up to time n, and q denotes the residual, i.e., $\eta(n) := \lfloor n/p \rfloor$ and $q := n - \eta p$.

Theorem 3. *Consider a robot moving in* $\mathbb{R}^2$ *whose trajectory is periodic with period p. In addition to Assumptions 1.1–4, assume that the first and second moments of the measurement errors are periodic with period p (so that* $\mathbf{b}_k = \mathbf{b}_{k+p}$*,* $\boldsymbol{\rho}_k = \boldsymbol{\rho}_{k+p}$ *and* $\mathbf{P}_k = \mathbf{P}_{k+p}$*). In that case,*

$$\begin{aligned}\mathrm{E}[\mathbf{e}(n)] &= \mathbf{t}^0_{0,q} - \left(I - (c\underline{\mathbf{R}})^p\right)^{-1} \\ &\quad \times \left(I - (c\underline{\mathbf{R}})^{\eta p}\right) w(p) - (c\underline{\mathbf{R}})^{\eta p} w(q),\end{aligned} \tag{19}$$

where $w(j)$ *is given by*

$$w(j) := \sum_{i=0}^{j-1}\left(c\underline{\mathbf{R}}\right)^i\mathbf{R}^0_{i+1}\left(c\underline{\mathbf{R}}\,\mathbf{t}^{i+1}_{i,i+1} + \boldsymbol{\rho}_i\right),$$

where $c, \underline{\mathbf{R}}$ *are as defined in Theorem 2.*

The proof of the theorem is provided in the Appendix. The assumption of the moments ρ_k etc. being periodic with period p is motivated by the use of vision-based sensors to measure relative poses. In that case the measurement error statistics may depend on the scene the camera sees, which will repeat itself every p instants due to the periodic nature of the robot's motion. Note that i.i.d. errors are a special case of errors with periodic statistics, so the result also holds if all the measurement errors are i.i.d.

It can be shown in a straightforward manner from (19) that the bias is $O(1)$, by using the fact that $|c| < 1$. This is consistent with Theorem 1, since the robot stays in a bounded region for all time when following a periodic trajectory.

4. Simulation verification

In this section we empirically estimate the mean and covariance of the estimation error by conducting Monte Carlo simulations and compare them with the theoretical predictions. In Section 4.1, we simulate a robot moving along a randomly generated three-dimensional path and compare the results with the upper and lower bounds predicted in Theorem 1. In Sections 4.2 and 4.3, we present simulations for the two-dimensional scenario with straight-line and periodic trajectories so that empirical results can be compared with predictions of Theorems 2 and 3. Here the robot is simulated moving along either the straight-line or periodic trajectory at a speed of 0.32 m/s for about 5.5 h, traveling a distance of 6400 m. In all three simulations, measurements of the robot's relative pose were taken every 0.2 s. All simulations are conducted in MATLAB©. To the extent possible, the parameters used in the simulations are the same as those in the experiments.

4.1. Three-dimensional simulation

For the three-dimensional case we simulate a robot moving along a path that is shown in Fig. 2. The robot traverses this path from the starting point to the left and moving to the right. Measurement errors are generated as follows. The error in rotation ($\tilde{\mathbf{R}}_k^{k-1}$) is introduced by applying a random unit-quaternion at each time step drawn independently from a Von Mises–Fisher distribution with concentration parameter value of 10 000. The reader is referred to [25] for details of the Von Mises–Fisher distribution. The errors in relative translation at each time step ($\tilde{\mathbf{t}}_{k-1,k}^k$) are drawn from a zero-mean normal random variable with covariance matrix $(2.5 \times 10^{-5})\, I_{3\times 3}$. The corresponding constants necessary to compute the upper bounds in Theorem 1 are obtained from randomly generated measurements to simulate a sensor characterization test, and are found to be $\gamma = 0.9997$, $\tau = 0.1295$ m, $b = 0$ m, $\beta = 0.008$ m, $\underline{p} = 7.45 \times 10^{-5}$ m^2, and $\overline{p} = 7.55 \times 10^{-5}$ m^2.

Fig. 3 compares the empirically estimated bias and variance with the upper bounds given by Theorem 1. The empirical estimates are obtained from 4500 Monte Carlo simulations. As predicted by the theorem, the bias in the position estimate grows without bound since the robot's position is growing (in norm) without bound. We see that the bounds predicted by the theorem are of the same order of magnitude as values obtained empirically. However, the bound for the variance is rather loose.

4.2. Straight-line two-dimensional trajectory

For the straight-line case, we simulate a robot moving in a straight line on a plane with a constant velocity of $[0.2263, 0.2263]^T$ m/s and constant orientation. Two types of simulation are conducted.

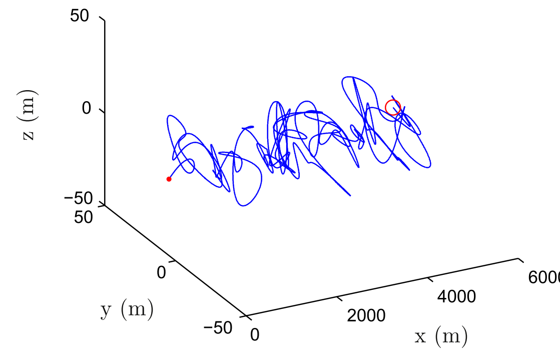


Fig. 2. The three-dimensional path used for the simulation in Section 4.1. The red dot indicates the robot's initial location, while the red circle indicates the robot's final location. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

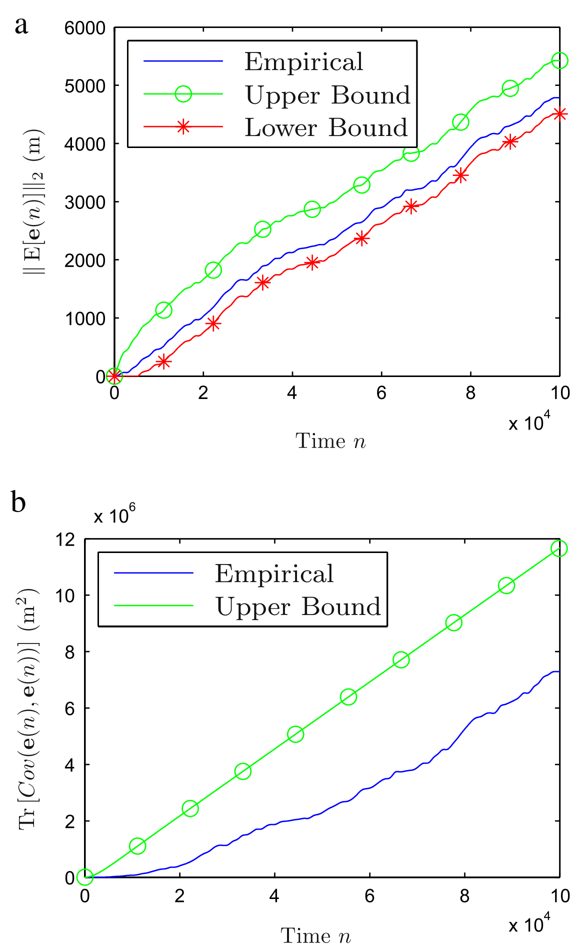


Fig. 3. Three-dimensional scenario: Comparison of Theorem 1's predictions ("Upper Bound" and "Lower Bound") of bounds on the bias and variance in position estimation error with those estimated from Monte Carlo simulations ("Empirical").

In the first type, which we call *simulated data*, noisy measurements of the rotation, i.e., $\hat{\theta}_{k-1,k}$, are generated as a Laplace-distributed random variable using a pseudo-random number generator. The reason for choosing a Laplace distribution over, say, a Gaussian, is the following. We obtained a large sample of two-dimensional orientation estimates from images taken with a machine-vision camera and performed hypothesis testing for three distributions: Laplacian, Gaussian and Fisher–Von Mises. Only the Laplace distribution passed the test. We refrain from giving details of the hypothesis testing here; they can be obtained from the authors upon request. Similar results about localization error also are obtained if a Gaussian distribution is used in generating two-dimensional rotation measurements, though we do not present them here. Both Gaussian and Laplacian distributions satisfy the requirement of Theorem 2, that of symmetry around the mean. Noisy measurements of the translations, i.e., $\hat{\mathbf{t}}^k_{k-1,k}$, were generated from noisy measurements of the translation direction, which we call $\boldsymbol{\zeta}^k_{k-1,k}$, and the translation magnitude, which we call $d_k \in \mathbb{R}^+$, as $\hat{\mathbf{t}}^k_{k-1,k} = \hat{d}_k \hat{\boldsymbol{\zeta}}^k_{k-1,k}$, where $\hat{d}_k$ and $\hat{\boldsymbol{\zeta}}^k_{k-1,k}$ are noisy estimates of d_k and $\boldsymbol{\zeta}^k_{k-1,k}$, respectively. Note that $\boldsymbol{\zeta}^k_{k-1,k}$ is a 2-vector with unit norm. This is done to simulate relative pose measurement with IMU/wheel odometry and a monocular camera without scale information. The camera provides the relative translation direction but not the magnitude of the translation, which is measured by IMUs/wheel encoders.

In the second type of simulation, which we call *simulated camera*, the vision-based relative pose estimation sensor is simulated in a more realistic fashion by generating synthetic image data, from which the relative rotation and direction of translation are estimated. The magnitudes of the translation measurements are generated as in the "simulated data" case.

Simulated data. At each time step k, a measurement of the relative orientation is constructed numerically as $\hat{\theta}_{k-1,k} = 0 + \tilde{\theta}_{k-1,k}$, where the orientation error $\tilde{\theta}_{k-1,k}$ is chosen to be a 0-mean Laplace-distributed random variable. Recall that a Laplace distribution with μ mean and variance $2\lambda^2$ has pdf $f(\tilde{\theta}) = \frac{1}{2\lambda} e^{-|\tilde{\theta}-\mu|/\lambda}$. The value of λ chosen is 3.6×10^{-3}, which best fits the orientation measurement error statistics generated by the synthetic monocular camera-based relative pose sensor that is used in the experiments described in what follows. The noisy measurement of translation direction $\hat{\boldsymbol{\zeta}}^k_{k-1,k}$ is generated as

$$\hat{\boldsymbol{\zeta}}^k_{k-1,k} = \begin{pmatrix} \cos\tilde{\phi}_{k-1,k} & -\sin\tilde{\phi}_{k-1,k} \\ \sin\tilde{\phi}_{k-1,k} & \cos\tilde{\phi}_{k-1,k} \end{pmatrix} \boldsymbol{\zeta}^k_{k-1,k},$$

where $\tilde{\phi}_{k-1,k}$ is a zero-mean Laplace random variable with variance 3.07×10^{-2} rad^2, and $\boldsymbol{\zeta}^k_{k-1,k} = \frac{1}{\sqrt{2}}[1, 1]^T$ is the true translation direction. The magnitude of the translation is $d_k = 6.4 \times 10^{-2}$ m and its noisy measurement is generated as $\hat{d}_k = d_k + \tilde{d}_k$, where $\tilde{d}_k$ is a zero-mean Gaussian random variable with mean 0 and variance 8.5467×10^{-5} m^2. These numbers are chosen to be consistent with those seen in an experiment with a wheeled robot described later in Section 5. The parameters $\mathbf{b}, c, \mathbf{P}, \boldsymbol{\rho}$, which are needed to compute the predictions by Theorem 2, are estimated by a simulated sensor characterization test, i.e., by appropriate averaging of randomly generated data. They turn out to be $\mathbf{b} = [-0.6842, -0.6842] \times 10^{-3}$ m, $c = 1 - 1.2873 \times 10^{-5}$, $\mathrm{Tr}[\mathbf{P}] = 1.2479 \times 10^{-4}$ m^2, and $\boldsymbol{\rho} = c\mathbf{b}$.

The mean and covariance of the position estimation error at every time instant are empirically estimated by averaging over 76,600 Monte Carlo simulations. Fig. 4 presents the estimated mean and covariances, and the values predicted by Theorem 2. We see from the figure that the prediction from Theorem 2 matches estimates from Monte Carlo simulations quite closely, even for the large time intervals used in the simulations.

Simulated camera. We now simulate the scenario in which relative pose measurements are obtained by a calibrated monocular Prosilica EC 1020 camera and wheel odometers found on a Pioneer P3-DX. To simulate an estimate of the camera ego-motion between consecutive time steps, suppose between k and $k+1$, a set of 50 three-dimensional points is randomly generated in the volume visible to the camera at time step k, with their coordinates represented in the coordinate frame attached to the camera at time step k. The points are then acted on by the true transformation from k to

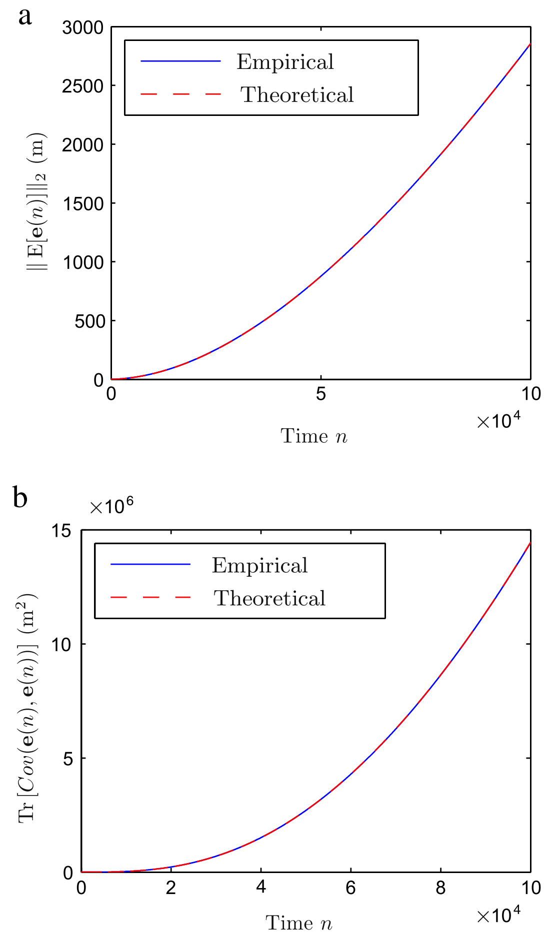
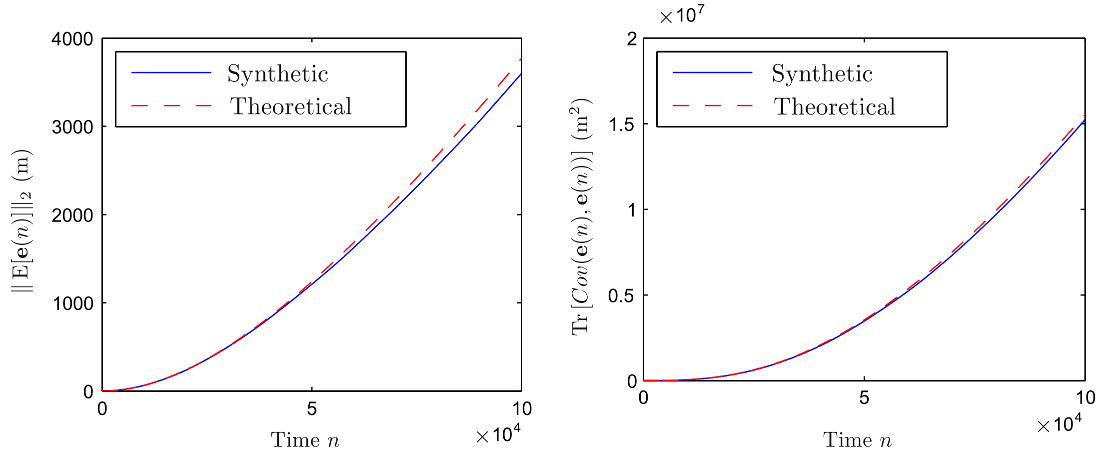


Fig. 4. Two-dimensional scenario, straight-line trajectory. Comparison of Theorem 2's predictions ("Theoretical") of the bias and variance in position estimation error with those obtained from Monte Carlo simulations ("Empirical"), for the "simulated data" case.

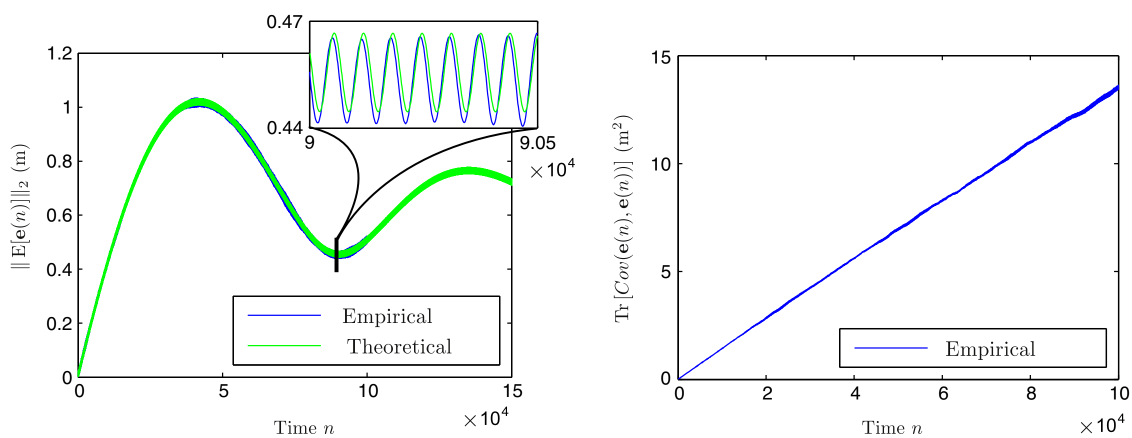
$k+1$ to find the corresponding coordinates in the coordinate frame attached to the camera at time step $k+1$, discarding any points falling outside the volume visible to the camera at that time step. Using a calibration matrix corresponding to the Prosilica EC 1020 camera, the points are projected into their corresponding image plane. This forms a set of correspondences analogous to the feature points extracted from actual image pairs. Each feature point is now corrupted by uniform noise with support lying in a 2×2 pixel square about the point. A RANSAC [26] assisted normalized 8-point algorithm [27] is used to estimate the rotation $\hat{R}$ and translation direction $\hat{\zeta}$ between the two time steps from these point correspondences. The axis of rotation was then aligned with the normal to the plane of motion and the component of the translation vector in that direction was dropped to ensure that the motion estimates remained in the plane. The magnitude of translation $\hat{d}$ is generated as in the *simulated data* case. The values of the parameters that are needed to compute the predictions by Theorem 2 are estimated from a simulated sensor characterization test like before. The values are found to be $\mathbf{b} = [-0.5767, -0.5904] \times 10^{-5}$ m, $\mathrm{Tr}[P] = 1.6382 \times 10^{-4}$ m^2, and $c = 1 - 2.1462 \times 10^{-5}$.

Fig. 5 compares the predictions of bias and variance by Theorem 2 to those estimated from 1000 Monte Carlo simulations. The number of Monte Carlo simulations is smaller in the synthetic data case due to the prohibitively high cost of conducting these simulations. We see from Fig. 5 that Theorem 2 accurately predicts the position estimation error computed from synthetic image data. The prediction toward the end of the simulation time is not as



(a) Bias. (b) Variance.

Fig. 5. Two-dimensional scenario, straight-line trajectory. Comparison of Theorem 2's predictions ("Theoretical") of the bias and variance in position estimation error with those obtained from Monte Carlo simulations ("Synthetic"), for the "simulated camera" case.



(a) Bias. (b) Variance.

Fig. 6. Two-dimensional scenario, periodic motion. Comparison of Theorem 3's predictions with estimates from Monte Carlo simulations ("Empirical"). The legend "Theoretical" in (a) refers to the prediction from (19) in Theorem 3.

accurate as in the simulated data case, which is due to the smaller number of Monte Carlo trials.

4.3. Periodic trajectory

We now simulate a robot moving on a circle with circumference of 4.11 m so that its trajectory is periodic with period $p = 3020$. The speed of the robot is approximately 0.32 m/s, so that it traverses the circle about 47 times before completing one period. The trajectory is chosen to be close to that encountered in an experiment with a Pioneer P3-DX robot, which will be described in Section 5. Noisy relative pose measurements are generated as in the *simulated data* case in straight-line motion. Orientation measurement errors are Laplace distributed (with mean $\mathrm{E}[\tilde{\theta}] = 6.8 \times 10^{-5}$ m and parameter $\lambda = 3.6 \times 10^{-3}$), while translation measurement errors are generated in the same manner, and with the same distributions as in the *simulated data* case in straight-line motion, with the new true values given by $\zeta^k_{k-1,k} = -[0.049, 0.999]^T$ and $d_k = 0.064$ m.

Fig. 6 shows the empirical estimates of bias and variance from 29,970 Monte Carlo simulations. It also presents the bias predicted from Theorem 3. We see from Fig. 6(a) that the bias is quite accurately predicted by Theorem 3. The high-frequency oscillation corresponds to the time it takes for the robot to traverse the circle once. The lower-frequency oscillation corresponds to the period of the trajectory. The variance seems to grow linearly with time, as one can see from Fig. 6(b), but a formula is not available in the periodic case for comparison.

5. Experimental verification

In this section, we report results of experiments conducted with a wheeled Pioneer P3-DX robot that is equipped with a calibrated monocular Prosilica EC 1020 camera and wheel odometers. The images captured by the camera are used to estimate the relative rotation and direction of translation. The distance traveled estimated by the wheel odometers is fused with the direction of translation estimated from the camera to estimate the translation vector. The relative pose of the camera is measured every 0.2 s. An overhead camera is used to measure the true two-dimensional pose of the robot. Due to space constraints of the indoor test setup, the trajectory of the robot was chosen to be an approximately circular one with radius 0.65 m and one rotation taking approximately 13 s (see Fig. 7). Although the robot's trajectory is not truly periodic, it is approximately periodic, with period $p = 3020$ (i.e., 604 s).

5.1. Test setup

Fig. 8(b) shows a schematic of the experimental setup. The global coordinate frame is defined to coincide with the coordi-

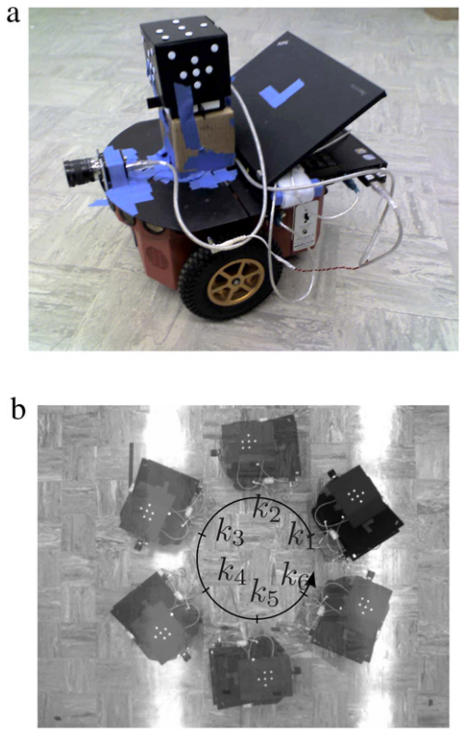


Fig. 7. (a) The robot used in the experiments, and (b) a few snapshots from the overhead camera showing the trajectory.

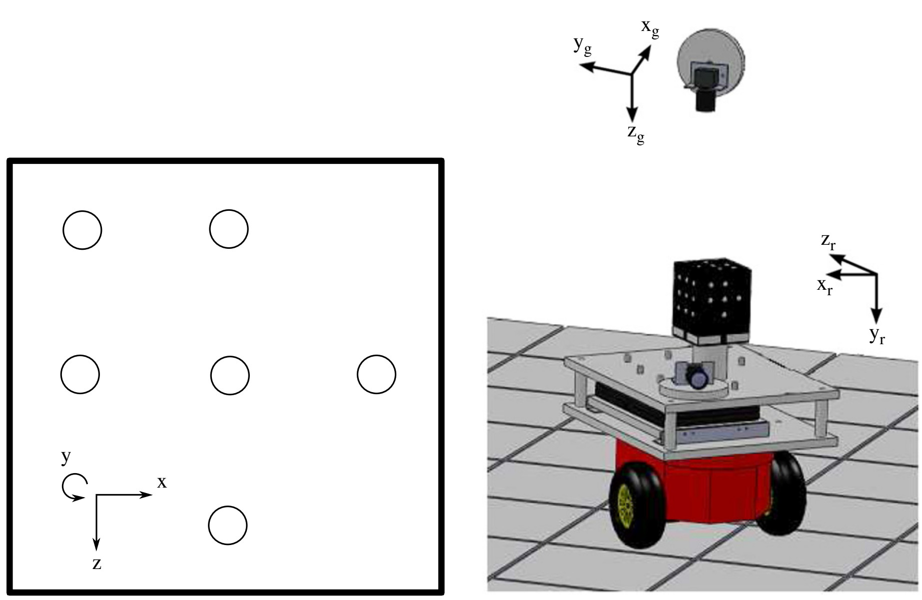
nate frame attached to an overhead camera viewing the plane of motion. That is, the origin of the global coordinate axes corresponds to the camera's focal point. The overhead camera is used to obtain the true pose of the robot. The robot's local coordinate frame was defined by a cube affixed to the top of the box. A grid consisting of six dots was placed atop the cube with a known geometry (see Fig. 8(a)), which allows reconstruction of the full three-dimensional pose of the robot from the single monocular camera. Although some error between the true pose of the robot and that estimated from the overhead camera is inevitable, this error did not have any cumulative effect over time. Therefore the pose estimated from the overhead camera is taken as the ground truth.

A KLT tracker [28] was used to track feature points across pairs of images, and a RANSAC-assisted normalized 8-point algorithm was used to estimate the relative rotation and direction of translation between every successive pairs of images. All estimation was performed off-line. Even with RANSAC, outliers in point correspondences can cause large errors in the relative pose estimates. An ad hoc "filter" was implemented to reduce the effect of such errors, as follows. If the estimated relative pose from the camera was deemed infeasible (which was determined by the known motion of the robot), the relative rotation and relative translation direction estimated in the previous time step were used as the estimate for the current time step. The relative translation between two time instants was estimated from the relative translation direction and the estimate of its magnitude, the latter being obtained from a wheel odometer. The relative poses so obtained were chained together to obtain an estimate of the global position and orientation of the robot at every time step, as described in Section 2.

5.2. Test results

The position estimation error at each time step is computed by comparing the ground truth with the robot's position estimated from relative pose measurements. Fig. 9 shows one instance of the true and estimated path. The bias and variance in the position estimation error at any given time step are determined by averaging over 17 experiments, where each experiment consists of the robot moving on its path for 1000 s (5000 time steps).

The experimentally obtained bias and variance of position estimation error are shown in Fig. 10(a) and (b). We see from the figures that the experimentally obtained results – especially the bias – closely resemble those seen in simulations (see Fig. 6(a) and (b)), which in turn are accurately predicted from the analysis. The experimentally obtained bias stays bounded, as Theorem 3 predicts. The variance also shows an on-average linear growth with time, which is consistent with Theorem 1. The experiment provides additional confidence in our theoretical results. In addition, we note that, while the theoretical predictions are for a dead-reckoning- type position estimation algorithm, the algorithm used



(a) Top view. (b) Coordinate axes.

Fig. 8. Schematic of the test setup.

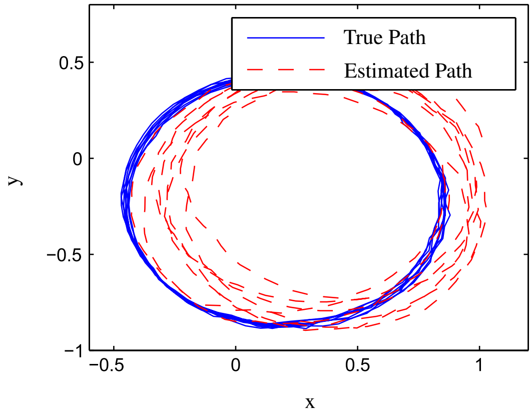


Fig. 9. A sample experiment with a P3-DX robot, showing the estimated and true positions.

in the experiments was more akin to a kinematic-model-based filter. Still, the theoretical predictions match the experimental results rather well. This is expected since – as argued earlier – the analysis is applicable to a broader class of estimation algorithms; see the discussion in Section 2 after Eq. (5).

There are nevertheless some discrepancies between the experimentally obtained bias and variance values and those obtained from simulations, as can be seen comparing Fig. 6(a) with Figs. 10(a) and 6(b) with Fig. 10(b). These are due to the differences between the experiments and simulations. First, the experimental bias and variances values are computed by averaging over *only* 17 experiments, whereas the simulation estimates are computed from at least 1000 Monte Carlo simulations, in some cases many more. The reason for this smaller number of experimental trials is the difficulty and time needed in performing these experiments. The smaller number of trials used for averaging produced less accurate estimates. Second, the characteristics of the camera error could not be modeled in any of our simulations. Third, it is not possible to ensure a truly periodic trajectory in a real experiment. The "high-frequency" oscillations in the experimental bias and variance plots are at 7.8×10^{-2} Hz, which correspond to the average time the robot takes to traverse the circle once. These are seen in the simulations as well; see in particular the inset in Fig. 6(a). However, these oscillations are not particularly visible in the variance; one has to magnify the curve in Fig. 6(b) considerably to see them. We believe the noticeable difference in case of the variance comes from the very small number of runs that we averaged over.

6. Reducing the bias

We now discuss a possible way to reduce the bias in the position estimate by using the lessons learned from the analysis that led to Theorem 1. First of all we note that computing the bias in the translation estimation error, i.e., $\mathrm{E}[\hat{\mathbf{t}}^0_{k-1,k}] - \mathbf{t}^0_{k-1,k}$, requires knowledge of true relative rotations and translations; see the expressions after (11). Therefore the bias in localization error cannot be eliminated by simply computing the bias in the translation estimation error and subtracting it from the estimated translation $\hat{\mathbf{t}}^0_{k-1,k}$ at every k. Instead the proposed method consist of modifying the *raw* measurements $\hat{\mathbf{R}}^{k-1}_k$, $\hat{\mathbf{t}}^k_{k-1,k}$ into the so-called *modified* measurements $(\hat{\mathbf{R}}^{k-1}_k)_{\text{modif}}$, $(\hat{\mathbf{t}}^k_{k-1,k})_{\text{modif}}$, that are defined below, and then using them in the position estimation.

$$(\hat{\mathbf{R}}^{k-1}_k)_{\text{modif}} := \hat{\mathbf{R}}^{k-1}_k(\bar{R})^{-1} \qquad (\hat{\mathbf{t}}^k_{k-1,k})_{\text{modif}} := \hat{\mathbf{t}}^k_{k-1,k} - \mathbf{b},$$

where

$$\bar{R} := \mathrm{E}[\tilde{\mathbf{R}}], \tag{20}$$

$$\mathbf{b} := \mathrm{E}[\tilde{\mathbf{t}}^k_{k-1,k}], \quad k \geq 1. \tag{21}$$

We are assuming that the translation measurements are stationary in mean so that $\mathbf{b}$ is a constant. The modified measurements can be computed from the raw measurements and knowledge of $\bar{R}$, $\mathbf{b}$, which can be determined from an analysis of sensor noise characteristics. For instance, the question of estimating $\mathbf{b}$ for vision-based sensors is examined in [13,12]. The position at time k is now computed as before, but with the new corrected measurements in place of the raw sensor measurements $\hat{\mathbf{t}}^k_{k-1,k}$ and $\hat{\mathbf{R}}^{k-1}_k$. Specifically,

$$(\hat{\mathbf{R}}^0_k)_{\text{modif}} := \prod_{i=1}^{k} (\hat{\mathbf{R}}^{i-1}_i)_{\text{modif}},$$

$$(\hat{\mathbf{t}}^0_{k-1,k})_{\text{modif}} := (\hat{\mathbf{R}}^0_k)_{\text{modif}}\,(\hat{\mathbf{t}}^k_{k-1,k})_{\text{modif}},$$

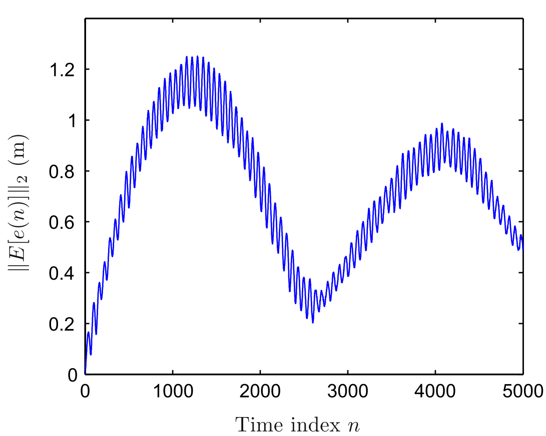
$$\text{and finally,} \quad (\hat{\mathbf{t}}^0_{0,n})_{\text{modif}} = \sum_{k=1}^{n} (\hat{\mathbf{t}}^0_{k-1,k})_{\text{modif}}.$$

The rationale for this proposal comes from the following relationships, which can be shown from straightforward calculations:

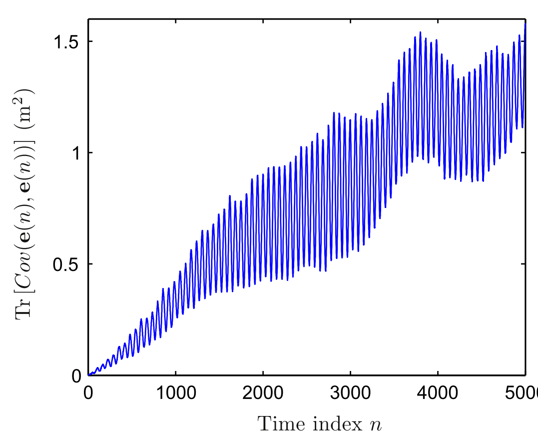
$$\mathrm{E}[(\hat{\mathbf{R}}^{k-1}_k)_{\text{modif}}] = \mathbf{R}^{k-1}_k \tag{22}$$

$$\mathrm{E}[(\hat{\mathbf{R}}^{k-1}_k)_{\text{modif}}(\hat{\mathbf{t}}^k_{k-1,k})_{\text{modif}}] = \mathbf{R}^{k-1}_k \mathbf{t}^k_{k-1,k}, \tag{23}$$

where the second relation (23) holds if the raw rotation and translation measurements $\hat{\mathbf{R}}^{k-1}_k$, $\hat{\mathbf{t}}^k_{k-1,k}$ are uncorrelated.



(a) Bias.



(b) Variance.

Fig. 10. Experimental results: bias and variance of position estimation error for a P3-DX robot (5000 time steps = 16.67 min).

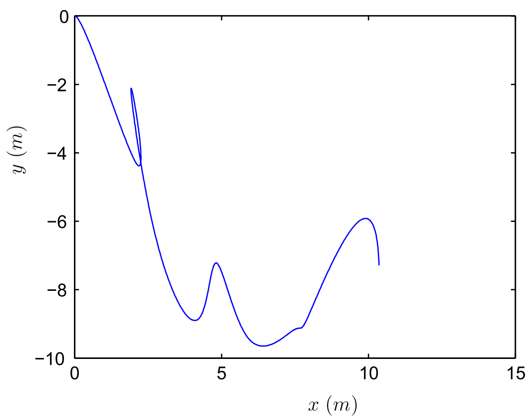


Fig. 11. A randomly generated path in two dimensions used to test the bias-reduction method.

The modification of the raw measurements, especially that of the rotation measurement, eliminates the geometric decay of the length of the relative translation measurements after being transformed to frame 0. As discussed in Section 3.2, this decay was the main cause of the bias growth. If $\hat{\mathbf{R}}_k^{k-1}$, $\hat{\mathbf{t}}_{k-1,k}^{k}$ are correlated but the motion is limited to a two-dimensional space, a slightly different method can be used that ensures that (22), (23) hold. The details are not provided, in the interest of saving space.

The proposed method was tested with the help of simulations to determine its effectiveness. The following types of trajectory in two dimensions were used in the simulations: (i) straight line (ii) circular, (iii) random walk in a city-like grid, and (iv) a randomly generated smooth path. The performance was seen to be similar in all cases; so we only present the details for the last case. The path our robot traversed in that case is shown in Fig. 11. Noise in the sensor measurements was simulated by adding i.i.d. Gaussian random vectors with mean $[0.05, 0.02]^T$ m and covariance matrix $0.05I$ to the relative translation measurements at each time step. The angle describing the relative rotation between each time step was corrupted by adding i.i.d. Gaussian random variables with mean 6.8×10^{-3} and variance 2.6×10^{-3}. The sensor characteristics $\overline{R}$ and $\mathbf{b}$ needed for the correction were determined a priori; their values are $\overline{R} = 0.9987, f_R(6.8 \times 10^{-3})$, $\mathbf{b} = [0.05, 0.02]^T$. The estimates of the bias and variance in the position estimates were obtained from more than a million Monte Carlo simulations. The comparison between the bias with the method described in Section 6 and that for the baseline case (no modification) is shown in Fig. 12(a). The comparison of the variances is shown in Fig. 12(b).

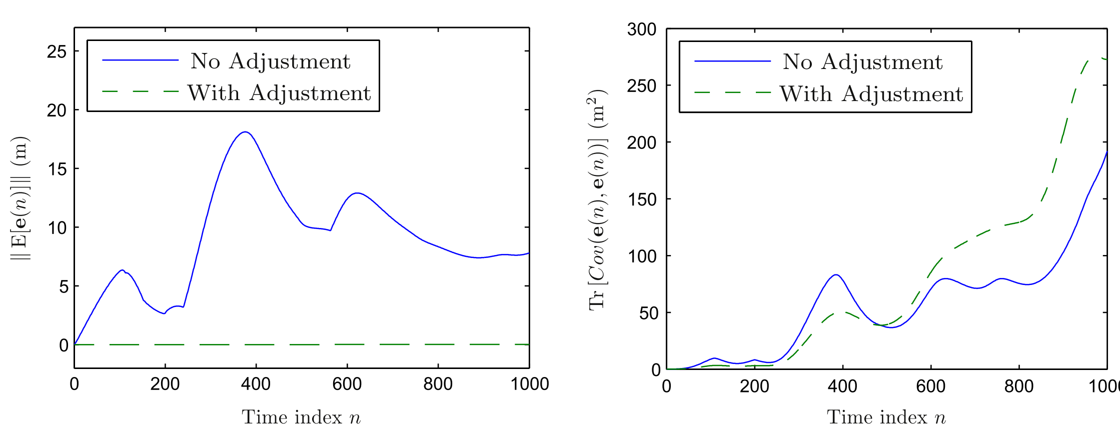
We see from the simulations that the proposed method significantly reduces the bias. The resulting variance is the same or smaller, for *small values of time*. For large values of time, the resulting variance is larger than that achieved if the measurements were not modified. This is expected, since the modifications introduce additional uncertainty. In particular, the modified rotation measurements are no longer elements of $SO(d)$. A similar trend is seen for all other trajectories tested: the bias is significantly reduced for all values of time, while the variance is either smaller or almost the same for small values of time but is larger for large values of time.

7. Summary

We examined the growth of error in position estimates obtained from noisy relative pose measurements. Asymptotic and non-asymptotic growth rates of the bias and variance of the error (with respect to time) were obtained. In both two dimensions and three dimensions, the bias and the variance of the position estimation error grows at most linearly with time or distance traveled. The variance growth rate is also lower bounded by a linear function of time if the translation measurement errors are large enough. Exact formulas for the error bias and variance were obtained for two special two-dimensional trajectories, straight line and periodic. Extensive Monte Carlo simulations, and experiments with a wheeled robot, were used to verify the results.

One of the assumptions made for the analysis was that the measurements collected at two distinct time instants are statistically independent. Though this may not hold in practice, the results obtained from experiments and simulations with synthetic image data are consistent with the theoretical predictions. This shows that the analysis is not sensitive to the assumptions of independence. The sufficient condition (10) for the variance to be asymptotically linear in time is not satisfied in the simulations and the experiment. However, the empirically estimated variance from simulations and experiment seems to grow linearly with time. This indicates that the sufficient condition is conservative. Determining a necessary condition for variance growth to be linear is an open question.

The precise growth rate of the bias depends on the trajectory of the robot. Specifically, if the robot stays in a bounded region, the bias is upper bounded by a constant for all time. The bias growth is principally determined by the fact that the expected value of the estimated position converges to a point, irrespective of how the robot is moving. This occurs since γ, the norm of the expected rotation error, is strictly less than unity. As a result, the magnitude of the measured relative translation (between two successive time



(a) Bias.

(b) Variance.

Fig. 12. Performance of the bias-reduction method, for the path shown in Fig. 11. The legend “With Adjustment” refers to the estimates obtained with the bias-reduction method of Section 6. The bias is reduced to almost zero with the proposed method. All quantities are estimated from more than a million Monte Carlo simulations.

instants), once the measurement is transformed to the global coordinate frame, decays geometrically with time. It turns out that the asymptotic growth rate of the bias does not change even if all the measurements are unbiased or even if the translation measurements are completely error free. An implication of the results is that, though the magnitude of the translation vector is an important determinant of the bias, the distance traveled is immaterial in determining either the bias or the variance.

A method to reduce the bias growth rate was suggested by the lessons learned in the analysis of error growth. Simulations showed that the proposed method reduces the bias significantly for all time, while having negligible effect on the variance for small values of time. Thus the method can be potentially used to improve localization accuracy for short periods of time. There are several issues that still need to be addressed. The method was observed to make the variance worse for large values of time. So an important research question is to determine the time period up to which the method can be used. The method requires knowledge of the sensor characteristics. Its robustness to imprecise knowledge of the sensor characteristics, and to time variations in those characteristics, also needs to be studied. Another line of research is to incorporate the proposed bias reduction method within a filtering-type position estimation algorithm.

Acknowledgments

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Appendix

Proof of Proposition 1. Let $\mathbf{y}$ be a d-dimensional random vector. Since $\mathrm{Cov}(\mathbf{y},\mathbf{y}) = \mathrm{E}[\mathbf{y}\mathbf{y}^T] - \mathrm{E}[\mathbf{y}]\mathrm{E}[\mathbf{y}]^T$, we have, upon taking the trace of both sides,

$$\|\mathrm{E}[\mathbf{y}]\|^2 = \mathrm{E}[\|\mathbf{y}\|^2] - \mathrm{Tr}\,[\mathrm{Cov}(\mathbf{y},\mathbf{y})] \le \mathrm{E}[\|\mathbf{y}\|^2],$$

since $\mathrm{Tr}\,[\mathrm{Cov}(\mathbf{y},\mathbf{y})] \ge 0$. Moreover, equality in the above inequality holds if and only if the variance of each of the components of $\mathbf{y}$ is 0, that is, $\mathbf{y}$ is degenerate. We now apply this result to the random vector $\mathbf{y} := \mathbf{R}x$, where x is a deterministic d-dimensional vector while $\mathbf{R}$ is a random rotation matrix:

$$\|\mathrm{E}[\mathbf{R}]x\|^2 \le \mathrm{E}[\|\mathbf{R}x\|^2] = \mathrm{E}[\|x\|^2] = \|x\|^2, \tag{24}$$

where the first equality is due to the fact that rotation does not change the 2-norm of a vector, and the second equality is due to x being deterministic. This proves that $\|\mathrm{E}[\mathbf{R}]\| \le 1$. Since $\mathbf{y}$ is degenerate if only if $\mathbf{R}$ is, the inequality in (24) is strict if $\mathbf{R}$ is non-degenerate. This proves the result. □

The following additional technical result is needed for the proof of Lemma 1.

Proposition 2. *If X_i is a sequence of random vectors such that $\mathrm{E}[X_i^TX_j] \le \overline{\alpha_0}\eta^{|i-j|}$, where $|\eta| < 1$ and $\overline{\alpha_0}$ is an arbitrary constant, then*

$$\mathrm{E}\left[\left(\sum_{i=1}^n X_i\right)^T\left(\sum_{i=1}^n X_i\right)\right] \le \overline{\alpha_0}\frac{1+|\eta|-2|\eta|^n}{1-|\eta|}n.$$

Furthermore, if $\underline{\alpha_0}\eta^{|i-j|} \le \mathrm{E}[X_i^TX_j]$ for $i \ne j$ and $0 < \underline{\beta_0} \le \mathrm{E}[X_i^TX_i]$, where $\underline{\alpha_0}$, $\underline{\beta_0}$ are constants such that $\underline{\beta_0} > 2\frac{|\underline{\alpha_0}|}{1-|\eta|}$, then $\mathrm{E}[(\sum_{i=1}^n X_i)^T(\sum_{i=1}^n X_i)] = \Theta(n)$.

Proof of Proposition 2. Expanding the sum, we obtain

$$\mathrm{E}\left[\left(\sum_{i=1}^n X_i\right)^T\left(\sum_{i=1}^n X_i\right)\right] = \sum_{i=1}^n T_i, \tag{25}$$

where

$$T_i := \sum_{j=1}^n \mathrm{E}[X_i^TX_j]. \tag{26}$$

It follows from (26) and the hypothesis that

$$\begin{aligned} T_i &\le \overline{\alpha_0}(\eta^{i-1}+\eta^{i-2}+\cdots+\eta+1+\eta+\cdots+\eta^{n-i}) \\ &\le \overline{\alpha_0}\left(-1+2\sum_{k=0}^{n-1}|\eta|^k\right) \\ &= \overline{\alpha_0}\frac{1+|\eta|-2|\eta|^n}{1-|\eta|}, \end{aligned}$$

where the second inequality follows from $1 \le i \le n$. The upper bound now follows from (25). This proves the first statement.

When the additional hypothesis holds, we have

$$\begin{aligned} T_i &\ge \underline{\alpha_0}(\eta^{i-1}+\eta^{i-2}+\cdots+\eta)+\underline{\beta_0}+\underline{\alpha_0}(\eta+\cdots+\eta^{n-i}) \\ &\ge -2|\underline{\alpha_0}|\sum_{k=0}^{\infty}|\eta|^k+\underline{\beta_0} = \underline{\beta_0}-2\frac{|\underline{\alpha_0}|}{1-|\eta|} =: \ell_0 > 0. \end{aligned}$$

It follows from (25) that $\mathrm{E}[(\sum_{i=1}^n X_i)^T(\sum_{i=1}^n X_i)] \ge n\ell_0 = \Omega(n)$. Combining the asymptotic lower and upper bounds, we get $\mathrm{E}[(\sum_{i=1}^n X_i)^T(\sum_{i=1}^n X_i)] = \Theta(n)$. □

Proof of Lemma 1. It follows from (4) that

$$\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}] = \sum_{k=1}^n \mathrm{E}[\hat{\mathbf{t}}^0_{k,k+1}]. \tag{27}$$

Recall that $\bar{R} = \mathrm{E}[\tilde{\mathbf{R}}]$. From (2)–(3), we get

$$\begin{aligned} &\hat{\mathbf{t}}^0_{k,k+1} = \mathbf{R}^0_1\tilde{\mathbf{R}}^0_1\cdots\mathbf{R}^k_{k+1}\tilde{\mathbf{R}}^k_{k+1}\left(\mathbf{t}^{k+1}_{k,k+1}+\tilde{\mathbf{t}}^{k+1}_{k,k+1}\right) \\ &\Rightarrow \mathrm{E}[\hat{\mathbf{t}}^0_{k,k+1}] = \mathbf{R}^0_k\bar{R}\cdots\mathbf{R}^k_{k+1}\left(\bar{R}\,\mathbf{t}^{k+1}_{k,k+1}+\boldsymbol{\rho}_{k+1}\right), \end{aligned}$$

where the second equality follows from the assumption that the orientation measurement errors are i.i.d. Since a rotation does not change the 2-norm of a vector,

$$\|\mathrm{E}[\hat{\mathbf{t}}^0_{k,k+1}]\| \le \|\bar{R}^k\|\left(\|\bar{R}\|\,\|\mathbf{t}^{k+1}_{k,k+1}\|+\|\boldsymbol{\rho}_{k+1}\|\right),$$

where the inequality follows from applying triangle inequality and using the sub-multiplicative property of induced norms. Since $\|\bar{R}^k\| \le \|\bar{R}\|^k$, we obtain upon using Proposition 1 and the definition of γ that

$$\|\mathrm{E}[\hat{\mathbf{t}}^0_{k,k+1}]\| \le \gamma^k a,$$

where $a := \sup_k(\|\bar{R}\|\,\|\mathbf{t}^{k+1}_{k,k+1}\|+\|\boldsymbol{\rho}_{k+1}\|) \le \gamma\tau+\beta$. Applying the triangle inequality to (27), we get

$$\|\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]\| \le \sum_{k=0}^{n-1}\|\mathrm{E}[\hat{\mathbf{t}}^0_{k,k+1}]\| \le a\sum_{k=0}^{n-1}\gamma^k \le a\frac{1-\gamma^n}{1-\gamma},$$

since $0 < \gamma < 1$. This proves the result about the mean.

The proof for the second moment result proceeds by first showing that $\mathrm{E}[(\hat{\mathbf{t}}^0_{j,j+1})^T\hat{\mathbf{t}}^0_{i,i+1}]$ satisfies the hypothesis of Proposition 2 and then applying the proposition. We note that, for $i \le j$,

$$\begin{aligned}(\hat{\mathbf{t}}^0_{i,i+1})^T\hat{\mathbf{t}}^0_{j,j+1} &= (\hat{\mathbf{t}}^{i+1}_{i,i+1}\hat{\mathbf{R}}^0_{i+1})^T\,\hat{\mathbf{R}}^0_{j+1}\hat{\mathbf{t}}^{j+1}_{j,j+1}\\ &= (\hat{\mathbf{t}}^{i+1}_{i,i+1})^T\,\hat{\mathbf{R}}^{i+1}_{j+1}\,\hat{\mathbf{t}}^{j+1}_{j,j+1}\\ &= V_1+V_2+V_3+V_4,\end{aligned}$$

where

$$\begin{aligned}V_1 &:= (\mathbf{t}^{i+1}_{i,i+1})^T\,\hat{\mathbf{R}}^{i+1}_{j+1}\,\mathbf{t}^{j+1}_{j,j+1}\\ V_2 &:= (\tilde{\mathbf{t}}^{i+1}_{i,i+1})^T\,\hat{\mathbf{R}}^{i+1}_{j+1}\,\mathbf{t}^{j+1}_{j,j+1}\\ V_3 &:= (\mathbf{t}^{i+1}_{i,i+1})^T\,\hat{\mathbf{R}}^{i+1}_{j+1}\,\tilde{\mathbf{t}}^{j+1}_{j,j+1}\\ V_4 &:= (\tilde{\mathbf{t}}^{i+1}_{i,i+1})^T\,\hat{\mathbf{R}}^{i+1}_{j+1}\,\tilde{\mathbf{t}}^{j+1}_{j,j+1}.\end{aligned}$$

We now evaluate the expected values of these four terms. By using the independence of the orientation measurement errors, we get

$$\begin{aligned}&\mathrm{E}[V_1] = (\mathbf{t}^{i+1}_{i,i+1})^T\,\mathbf{R}^{i+1}_{i+2}\overline{R}\cdots\mathbf{R}^{j}_{j+1}\overline{R}\,\mathbf{t}^{j+1}_{j,j+1}\\ &\Rightarrow |\mathrm{E}[V_1]| \le \|\mathbf{t}^{i+1}_{i,i+1}\|\,\|\overline{R}^{j-i}\,\mathbf{t}^{j+1}_{j,j+1}\|\\ &\qquad\qquad \le \|\overline{R}^{j-i}\|\,\|\mathbf{t}^{i+1}_{i,i+1}\|\,\|\mathbf{t}^{j+1}_{j,j+1}\| \le \gamma^{j-i}\tau^2,\end{aligned}$$

where the first inequality uses the fact that rotations do not change the 2-norm. For V_2, since $\tilde{\mathbf{t}}^{i+1}_{i,i+1}$ is statistically dependent only on $\tilde{\mathbf{R}}^i_{i+1}$ and not on $\tilde{\mathbf{R}}^{i+1}_{i+2},\dots,\tilde{\mathbf{R}}^j_{j+1}$, it is also independent of $\hat{\mathbf{R}}^{i+1}_{j+1}$. Hence,

$$|\mathrm{E}[V_2]| = |\mathbf{b}_i\,\mathbf{R}^{i+1}_{i+2}\,\overline{R}\mathbf{R}^j_{j+1}\,\overline{R}\mathbf{t}^{j+1}_{j,j+1} \Rightarrow |\mathrm{E}[V_2]| \le \gamma^{j-i}\,b\tau.$$

Similarly, we have, for $i < j$,

$$\begin{aligned}&\mathrm{E}[V_3] = (\mathbf{t}^i_{i+1,i+1})^T\,\mathbf{R}^{i+1}_{j+1}\,\overline{R}\,\mathbf{R}^{j-1}_j\overline{R}\,\boldsymbol{\rho}_{j+1}\\ &\Rightarrow |\mathrm{E}[V_3]| \le \gamma^{j-i}\frac{1}{\gamma}\tau\rho\end{aligned}$$

and, for $i = j$, $|\mathrm{E}[V_3]| \le \tau b$. For V_4, when $i < j$, we have

$$\begin{aligned}&V_4 = (\tilde{\mathbf{t}}^{i+1}_{i,i+1})^T\,\mathbf{R}^{i+1}_{i+2}\,\tilde{\mathbf{R}}^{i+1}_{i+2}\cdots\mathbf{R}^j_{j+1}\,\tilde{\mathbf{R}}^j_{j+1}\,\tilde{\mathbf{t}}^{j+1}_{j,j+1},\\ &\Rightarrow |\mathrm{E}[V_4]| \le \|\mathbf{b}_i\|\,\|\overline{\mathbf{R}}\|^{j-i-1}\|\boldsymbol{\rho}_{j+1}\| \le \gamma^{j-i}\frac{1}{\gamma}b\rho.\end{aligned}$$

When $i = j$, we have $V_4 = (\tilde{\mathbf{t}}^{j+1}_{j,j+1})^T\ \tilde{\mathbf{t}}^{j+1}_{j,j+1}$, which implies that $\mathrm{E}[V_4] = \mathrm{Tr}\left[\mathbf{P}_{j+1}\right] + \mathbf{b}^T_{j+1}\mathbf{b}_{j+1}$, by definition. Therefore,

$$0 < \underline{p} \le \mathrm{E}[V_4] \le \overline{p} + b^2. \quad (i = j).$$

Combining all four terms, we get

$$\begin{aligned}&-\alpha_0\gamma^{j-i} \le \mathrm{E}[(\hat{\mathbf{t}}^0_{i,i+1})^T\hat{\mathbf{t}}^0_{j,j+1}] \le \alpha_0\gamma^{j-i}, \quad (i < j)\\ &x_2 \le \mathrm{E}[(\hat{\mathbf{t}}^0_{i,i+1})^T\hat{\mathbf{t}}^0_{i,i+1}] \le x_1,\end{aligned}$$

where $\alpha_0 := \tau^2+\tau b+\frac{1}{\gamma}\tau\rho+\frac{1}{\gamma}b\rho$ and $x_1 := \tau^2+2\tau b+\overline{p}+b^2$, $x_2 := \max(\underline{p} - \tau^2 - 2\tau b, 0)$. Repeating these arguments for $i \ge j$ and combining, we find that

$$\underline{z_0}\gamma^{|i-j|} \le \mathrm{E}[(\hat{\mathbf{t}}^0_{i,i+1})^T\hat{\mathbf{t}}^0_{j,j+1}] \le \overline{z_0}\gamma^{|i-j|},$$

where $\overline{z_0} := \max\{z_0, x_1\}$, and $\underline{z_0} := \min(-\alpha_0, x_2) = -\alpha_0$. Now call $X_i := \hat{\mathbf{t}}^0_{i,i+1}$, so that $\hat{\mathbf{t}}^0_{0,n} = \sum_{i=0}^{n-1} X_i$. Hence, $\mathrm{E}[(\hat{\mathbf{t}}^0_{0,n})^T\hat{\mathbf{t}}^0_{0,n}] = \mathrm{E}[(\sum_{i=0}^{n-1} X_i)^T(\sum_{j=0}^{n-1} X_j)]$. It now follows from Proposition 2 that

$$\mathrm{E}[(\hat{\mathbf{t}}^0_{0,n})^T\hat{\mathbf{t}}^0_{0,n}] \le \overline{\alpha_0}\,\frac{1+\gamma-2\gamma^n}{1-\gamma}n.$$

This proves the first statement of the lemma. The upper bound is clearly $O(n)$.

It also follows from Proposition 2 that a lower bound on $\mathrm{E}[(\hat{\mathbf{t}}^0_{0,n})^T\hat{\mathbf{t}}^0_{0,n}]$ is $\Omega(n)$ if $\underline{\beta_0} > 2\frac{\alpha_0}{1-\gamma}$. Since $\alpha_0 = \tau^2 + \tau b + \tau\rho$, the condition $\underline{\beta_0} > 2\frac{\alpha_0}{1-\gamma}$ is equivalent to $\underline{p} > 2b\tau+\tau^2+2\frac{(\tau+\rho/\gamma)(\tau+b)}{1-\gamma}$, which proves the result. □

Proof of Theorem 2. Define a new random variable, $\delta\tilde{\theta}_{k-1,k} := \tilde{\theta}_{k-1,k} - \mathrm{E}[\tilde{\theta}_{k-1,k}]$. Then $\{\delta\tilde{\theta}_{k-1,k}\}_{k=0}^{\infty}$ is an i.i.d. sequence and the marginal density of $\delta\tilde{\theta}_{k-1,k}$ is symmetric about 0. We define the corresponding rotation matrices $\delta\tilde{R}^i_j := f_R(\delta\tilde{\theta}_{i,j})$. Utilizing the commutative property of two-dimensional rotation matrices, we have $\tilde{\mathbf{R}}^i_j = \left(\underline{\mathbf{R}}^{j-i}\right)\delta\tilde{R}^i_j$. It then follows from (5) that

$$\mathbf{e}(n) = n\mathbf{r} - \hat{\mathbf{t}}^0_{0,n},$$

and from (4), (3) and (2) that

$$\hat{\mathbf{t}}^0_{0,n} = \sum_{k=1}^{n}\left(\prod_{i=1}^{k}\underline{\mathbf{R}}\delta\tilde{R}^{i-1}_i\right)\ \left(\mathbf{r}+\tilde{\mathbf{t}}^k_{k-1,k}\right),$$

where we have used the fact that $\hat{\mathbf{R}}^{i-1}_i = \mathbf{R}^{i-1}_i\tilde{\mathbf{R}}^{i-1}_i = \underline{\mathbf{R}}\delta\tilde{R}^{i-1}_i$, since $\mathbf{R}^{i-1}_i = I$ due to the nature of the trajectory. We define two new random variables

$$\begin{aligned}\mathbf{f}_n &:= \sum_{k=1}^{n}\left(\prod_{i=1}^{k}\underline{\mathbf{R}}\delta\tilde{R}^{i-1}_i\right)\mathbf{r}\\ \mathbf{g}_n &:= \sum_{k=1}^{n}\left(\prod_{i=1}^{k}\underline{\mathbf{R}}\delta\tilde{R}^{i-1}_i\right)\ \tilde{\mathbf{t}}^k_{k-1,k},\end{aligned}$$

so that

$$\hat{\mathbf{t}}^0_{0,n} = \mathbf{f}_n + \mathbf{g}_n. \tag{28}$$

By the i.i.d. assumption on the sequence $\{\tilde{\theta}_{k-1,k}\}_k$, the sequence $\{\delta\tilde{R}^{k-1}_k\}_k$ is also i.i.d., so that

$$\mathrm{E}[\delta\tilde{R}^i_j] = \mathrm{E}\left[\prod_{k=i+1}^{k}\delta\tilde{R}^{k-1}_k\right] = \prod_{k=i-1}^{j}\mathrm{E}\left[\delta\tilde{R}^{k-1}_k\right] = c^{j-i}I, \tag{29}$$

where we have used the fact that $\mathrm{E}[\sin\delta\tilde{\theta}_{i-1,i}] = 0$, which follows from Assumption 1. It is then straightforward to show that

$$\begin{aligned}\mathrm{E}[\mathbf{f}_n] &= \sum_{k=1}^{n}(c\underline{\mathbf{R}})^k\,\mathbf{r} = \left(I - c\underline{\mathbf{R}}\right)^{-1}\left(I - (c\underline{\mathbf{R}})^n\right)c\underline{\mathbf{R}}\mathbf{r}\\ \mathrm{E}[\mathbf{g}_n] &= \sum_{k=0}^{n-1}(c\underline{\mathbf{R}})^k\,\boldsymbol{\rho} = \left(I - c\underline{\mathbf{R}}\right)^{-1}\left(I - (c\underline{\mathbf{R}})^n\right)\boldsymbol{\rho}.\end{aligned}$$

The expected value $\mathbf{e}(n)$ is now

$$\mathrm{E}[\mathbf{e}(n)] = n\,\mathbf{r} - \left(I - c\underline{\mathbf{R}}\right)^{-1}\left(I - (c\underline{\mathbf{R}})^n\right)\left(c\underline{\mathbf{R}}\mathbf{r}+\boldsymbol{\rho}\right), \tag{30}$$

which proves the first equality in (16).

For the variance, it follows from (28) that

$$\begin{aligned}\mathrm{Tr}\left[\mathrm{Cov}(\mathbf{e}(n),\mathbf{e}(n))\right] &= \mathrm{Tr}\left[\mathrm{Cov}(\hat{\mathbf{t}}^0_{0,n},\hat{\mathbf{t}}^0_{0,n})\right]\\ &= \mathrm{E}[\mathbf{f}^T_n\mathbf{f}_n] + \mathrm{E}[\mathbf{g}^T_n\mathbf{g}_n] + 2\,\mathrm{E}[\mathbf{f}^T_n\mathbf{g}_n]\\ &\quad - \mathrm{E}[\hat{\mathbf{t}}^0_{0,n}]^T\mathrm{E}[\hat{\mathbf{t}}^0_{0,n}].\end{aligned} \tag{31}$$

$$\mathrm{E}[\mathbf{f}_n^T\mathbf{f}_n] = \mathbf{r}^T\mathrm{E}\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{i}\underline{\mathbf{R}}\,\delta\tilde{R}_j^{j-1}\right)^T\sum_{k=1}^{n}\left(\prod_{\ell=1}^{k}\underline{\mathbf{R}}\,\delta\tilde{R}_\ell^{\ell-1}\right)\right]\mathbf{r}$$
$$= \mathbf{r}^T\Big[\left(I + c\underline{\mathbf{R}}^T + \cdots + (c\underline{\mathbf{R}}^T)^{n-1}\right) + \left(c\underline{\mathbf{R}} + I + c\underline{\mathbf{R}}^T + \cdots + (c\underline{\mathbf{R}}^T)^{n-2}\right)\cdots + \left((c\underline{\mathbf{R}})^{n-1} + \cdots + I\right)\Big]\mathbf{r},$$

where we have used the independence of the sequence $\{\delta\tilde{R}_k^{k-1}\}_k$ and the fact that $\delta\tilde{R}_k^{k-1}\delta\tilde{R}_k^{k-1} = I = \underline{\mathbf{R}}\,\underline{\mathbf{R}}^T$. The expression above simplifies to

$$\mathrm{E}[\mathbf{f}_n^T\mathbf{f}_n] = \mathbf{r}^T\left[nI + 2\sum_{k=1}^{n-1}(n-k)\left(c\underline{\mathbf{R}}\right)^k\right]\mathbf{r} = \mathbf{r}^T\left(I - c\underline{\mathbf{R}}\right)^{-2} \times \left(I + 2(n-2)c\underline{\mathbf{R}} - 2(n-1)\left(c\underline{\mathbf{R}}\right)^2 + 2\left(c\underline{\mathbf{R}}\right)^{n+1}\right)\mathbf{r}.$$

To examine $\mathrm{E}[\mathbf{g}_n^T\mathbf{g}_n]$, we express the product as $\mathbf{g}_n^T\mathbf{g}_n = \sum_{k=1}^{n} T_k$, where

$$T_k = (\tilde{\mathbf{t}}_{k-1,k}^k)^T\left((\delta\tilde{R}_k^{k-1})^T\,(\delta\tilde{R}_{k-1}^{k-2})^T\cdots(\delta\tilde{R}_1^0)^T\right)(\underline{\mathbf{R}}^k)^T \times\left(\underline{\mathbf{R}}\,\delta\tilde{R}_1^0\tilde{\mathbf{t}}_{0,1}^1 + \cdots + \underline{\mathbf{R}}^n\,\delta\tilde{R}_1^0\cdots\delta\tilde{R}_k^{k-1}\tilde{\mathbf{t}}_{k-1,k}^k\right).$$

Taking the expectation and using the assumptions on the noise correlations, we get, for $k > 1$,

$$\mathrm{E}[T_k] = \mathrm{Tr}\left[P + \mathbf{b}\mathbf{b}^T\right] + \mathbf{b}^T((c\underline{\mathbf{R}})^{k-2} + (c\underline{\mathbf{R}})^{k-3} + \cdots + I + I + (c\underline{\mathbf{R}}) + (c\underline{\mathbf{R}})^2 + \cdots + (c\underline{\mathbf{R}})^{n-1-k})\boldsymbol{\rho},$$

and for $k = 1$, $\mathrm{E}[T_k] = \mathrm{Tr}\left[\mathbf{P} + \mathbf{b}\mathbf{b}^{\mathbf{T}}\right] + \mathbf{b}^T(I + c\underline{\mathbf{R}} + (c\underline{\mathbf{R}})^2 + \cdots + (c\underline{\mathbf{R}})^{n-1-k})\boldsymbol{\rho}$. Repeating this for all the T_k, we get

$$\mathrm{E}[\mathbf{g}_n^T\mathbf{g}_n] = n\mathrm{Tr}\left[\mathbf{P} + \mathbf{b}\mathbf{b}^T\right] + \mathbf{b}^T\left[2\sum_{k=0}^{n-2}(n-k-1)\left(c\underline{\mathbf{R}}\right)^k\right]\boldsymbol{\rho}$$
$$= n\mathrm{Tr}\left[\mathbf{P} + \mathbf{b}\mathbf{b}^T\right] + \mathbf{b}^T\left(I - c\underline{\mathbf{R}}\right)^{-2} \times \left[2(n-1)I - 2nc\underline{\mathbf{R}} + 2\left(c\underline{\mathbf{R}}\right)^n\right]\boldsymbol{\rho}.$$

Similar tedious calculations lead to the following:

$$\mathrm{E}[\mathbf{f}_n^T\mathbf{g}_n] = \left[\sum_{k=0}^{n-1}\mathbf{b}^T(c\underline{\mathbf{R}})^k + \sum_{k=0}^{n-2}(n-k-1)\boldsymbol{\rho}^T(c\underline{\mathbf{R}}^T)^k\right]\mathbf{r}$$
$$= \mathbf{b}^T\left(I - c\underline{\mathbf{R}}\right)^{-2}\left[I - c\underline{\mathbf{R}} - \left(c\underline{\mathbf{R}}\right)^n + \left(c\underline{\mathbf{R}}\right)^{n+1}\right]\mathbf{r} + \mathbf{r}^T\left(I - c\underline{\mathbf{R}}\right)^{-2}\left[(n-1)I - nc\underline{\mathbf{R}} + (c\underline{\mathbf{R}})^n\right]\boldsymbol{\rho}.$$

Plugging all of this back in (31), we get $\mathrm{Tr}\left[\mathrm{Cov}\left(\mathbf{e}(n), \mathbf{e}(n)\right)\right] = \psi n + \omega(n)$, where ψ, $\omega(n)$ are given in (18), proving the second equality in (16). □

Proof of Theorem 3. Define a new random variable, $\delta\tilde{\theta}_{k-1,k} := \tilde{\theta}_{k-1,k} - \mathrm{E}[\tilde{\theta}_{k-1,k}]$. The sequence $\{\delta\tilde{\theta}_{k-1,k}\}_{k=0}^{\infty}$ is then i.i.d. and the marginal density of $\delta\tilde{\theta}_{k-1,k}$ is symmetric about the origin for each k. We define the corresponding rotation matrices $\delta\tilde{R}_j^i := f_R(\delta\tilde{\theta}_{i,j})$. Utilizing the commutative property of rotations in two dimensions, we have the following relation:

$$\tilde{\mathbf{R}}_j^i = \left(\underline{\mathbf{R}}^{j-i}\right)\delta\tilde{R}_j^i. \quad (32)$$

To examine the bias, we first rewrite the position estimate $\hat{\mathbf{t}}_{0,n}^0$ as

$$\hat{\mathbf{t}}_{0,n}^0 = \sum_{i=0}^{n}\hat{\mathbf{t}}_{i,i+1}^0 = \sum_{k=0}^{\eta-1}\left(\sum_{m=1}^{p}\hat{\mathbf{t}}_{kp+m-1,kp+m}^0\right) + \sum_{j=1}^{q}\hat{\mathbf{t}}_{\eta p+j-1,\eta p+j}^0, \quad (33)$$

where the first term is sum is over all time steps up to the end of the last (η-th) period and the second term is for the time steps after that. For any $0 \le m < p$, we have

$$\hat{\mathbf{t}}_{kp+m-1,kp+m}^0 = \hat{\mathbf{R}}_{kp+m}^0\,\hat{\mathbf{t}}_{kp+m-1,kp+m}^{kp+m} = \mathbf{R}_m^0\,\tilde{\mathbf{R}}_{kp+m}^0(\mathbf{t}_{m-1,m}^m + \tilde{\mathbf{t}}_{kp+m-1,kp+m}^{kp+m}),$$

where, apart from $\hat{R} = R\tilde{R}$, we have used the periodic nature of the trajectory that leads to $\mathbf{R}_{kp+m}^0 = \mathbf{R}_m^0$ and $\mathbf{t}_{kp+m-1,kp+m}^{kp+m} = \mathbf{t}_{m-1,m}^m$. Taking the expectation and using (32), we obtain

$$\mathrm{E}[\hat{\mathbf{t}}_{kp+m-1,kp+m}^0] = \mathbf{R}_m^0(c\underline{\mathbf{R}})^{kp+m-1}(c\underline{\mathbf{R}}\mathbf{t}_{m-1,m}^m + \boldsymbol{\rho}_m).$$

This expression is used to evaluate $\mathrm{E}[\mathbf{t}_{0,n}^0]$ by taking the expectation of the right-hand side of (33). After grouping terms, we obtain

$$\mathrm{E}[\mathbf{t}_{0,n}^0] = \left(\sum_{k=0}^{\eta-1}(c\underline{\mathbf{R}})^{kp}\omega(p)\right) + (c\underline{\mathbf{R}})^{\eta p}\omega(q). \quad (34)$$

Using techniques similar to those used in the proof of Theorem 2, it can be shown that

$$\mathrm{E}[\hat{\mathbf{t}}_{0,n}^0] = \sum_{i=0}^{\eta-1}(c\underline{\mathbf{R}})^{ip}w + (c\underline{\mathbf{R}})^{\eta p}w(q)$$
$$\Rightarrow \mathrm{E}[\mathbf{e}(n)] = \sum_{k=0}^{q-1}\mathbf{R}_{k+1}^0\mathbf{t}_{k,k+1}^{k+1} - \sum_{k=0}^{\eta-1}\left(c\underline{\mathbf{R}}\right)^{kp}w - (c\underline{\mathbf{R}})^{\eta p}w(q).$$

By replacing the summation we arrive at (19), which proves the theorem. □

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