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Approximation error in PDE-based modelling of vehicular platoons

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We study the problem of how much error is introduced in approximating the dynamics of a large vehicular platoon by using a partial differential equation, as was done in Barooah, Mehta, and Hespanha [Barooah, P., Mehta, P.G., and Hespanha, J.P. (2009), 'Mistuning-based Decentralised Control of Vehicular Platoons for Improved Closed Loop Stability', *IEEE Transactions on Automatic Control*, 54, 2100–2113], Hao, Barooah, and Mehta [Hao, H., Barooah, P., and Mehta, P.G. (2011), 'Stability Margin Scaling Laws of Distributed Formation Control as a Function of Network Structure', *IEEE Transactions on Automatic Control*, 56, 923–929]. In particular, we examine the difference between the stability margins of the coupled-ordinary differential equations (ODE) model and its partial differential equation (PDE) approximation, which we call the approximation error. The stability margin is defined as the absolute value of the real part of the least stable pole. The PDE model has proved useful in the design of distributed control schemes (Barooah et al. 2009; Hao et al. 2011); it provides insight into the effect of gains of local controllers on the closed-loop stability margin that is lacking in the coupled-ODE model. Here we show that the ratio of the approximation error to the stability margin is $O(1/N)$, where N is the number of vehicles. Thus, the PDE model is an accurate approximation of the coupled-ODE model when N is large. Numerical computations are provided to corroborate the analysis.

Keywords: error analysis; partial differential equation; continuum approximation; stability margin; eigenvalue approximation; distributed control

1. Introduction

Partial differential equations (PDEs) have been gaining attention in studying large-scale distributed systems such as power network, coupled-oscillators, vehicular platoons and formation control problems (Justh and Krishnaprasad 2003; Parashar, Thorp, and Seyler 2004; Sarlette and Sepulchre 2009; Frihauf and Krstic 2010; Hao and Barooah 2010; Yin, Mehta, Meyn, and Shanbhag 2010). A PDE approximation is frequently used in the analysis of many-particle systems in statistical physics and traffic-dynamics (see Helbing 2001) and the references therein. A similar but distinct framework based on partial *difference* equations defined on graphs has also been proposed in Trecate, Buffa, and Gati (2006) to study multi-agent coordination problems. In addition, there are also extensive literature studying the opposite problem, i.e. approximate the continuous system by discretisation. For example, Brockett and Willems (1974) examines control systems defined on modules by the discretisation of linear constant-coefficient PDEs. Another approach based on spatial Fourier transforms has been used in Bamieh, Paganini, and Dahleh (2002) to study infinite-dimensional quadratic optimal control problems for spatially invariant systems.

A typical question in continuum approximation of a discrete problem or discrete approximation of a continuum problem is the amount of error that is introduced by the approximation. A PDE-based methodology is proposed for the analysis and design of distributed control laws for a large vehicular formation in the recent work by Barooah, Mehta, and Hespanha (2009) and Hao, Barooah, and Mehta (2011). This article is on the error in the PDE approximation used in these references. The PDE model is obtained as a continuum approximation of the coupled-ODE model of the platoon when the number of vehicles, N, is large. The PDE provides insight into the effect of control architecture on the stability margin, where the stability margin is defined as the absolute value of the real part of the least stable closed-loop eigenvalue. Based on the PDE approximation of a platoon's dynamics, it is shown in Barooah et al. (2009) that *asymmetric* control architecture, in which a vehicle uses information from its front and back neighbouring vehicles differently, improves the stability margin over *symmetric* control architecture, in which information from both front and back vehicles are used with equal weight. More specifically, it is shown that with a symmetric control, the stability

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margin of the platoon scales as $O(1/N^2)$, whereas with an extremely small amount of front–back asymmetry in the control gains, this scaling can be improved to $O(\epsilon/N)$, where ϵ denotes the amount of asymmetry in the control gains. These scaling laws are obtained by analysing the stability margin of the PDE model. Numerical computation of the stability margin of the coupled-ODE model shows an excellent match between the prediction of the PDE model and that of the coupled-ODE model. These results are extended to the case of higher-dimensional formations (not merely one-dimensional platoons) in Hao et al. (2011).

The PDE approximation-based analysis carried out in Barooah et al. (2009) and Hao et al. (2011), however, does not answer the question of how large is the *approximation error*, which we define to be the difference between the stability margin of the PDE model and of the coupled-ODE model. This article addresses this question. Specifically, we show that the approximation error with symmetric control is $O(1/N^3)$ and with asymmetric control is $O(\epsilon/N^2)$. Comparing these errors to the stability margins themselves, we see that the ratio of approximation error to the stability margin itself is $O(1/N)$, which is negligible for a large value of N. This provides a rigorous justification of the PDE approximation used in Barooah et al. (2009) and Hao et al. (2011). The results in this article, and in fact those in Barooah et al. (2009) and Hao et al. (2011), are limited to the case when the asymmetry in control gains is sufficiently small. This limitation comes from the use of perturbation techniques to establish the results.

The rest of this article is organised as follows. Section 2 describes the coupled-ODE and PDE models and presents the main results. The proofs of the results appear in Sections 3 and 4, as well as numerical verification. This article ends with a summary in Section 5.

2. Models and main results

2.1 *Coupled-ODE and PDE models*

The underlying problem is that of the formation control of N homogeneous vehicles moving in a straight line, as shown in Figure 1(a). The position of each vehicle is denoted by $p_i \in \mathbb{R}$ ($i \in \{1, 2, \ldots, N\}$), and the dynamics of each vehicle are modelled as a double integrator: $\ddot{p}_i = u_i$, where u_i is the control input. The control objective is that vehicles maintain a desired formation geometry while following a constant-velocity-type desired trajectory. The information on the desired trajectory of the platoon is given in terms of a *fictitious* reference vehicle, which perfectly tracks its desired trajectory $p_0^*(t)$. The desired geometry of the

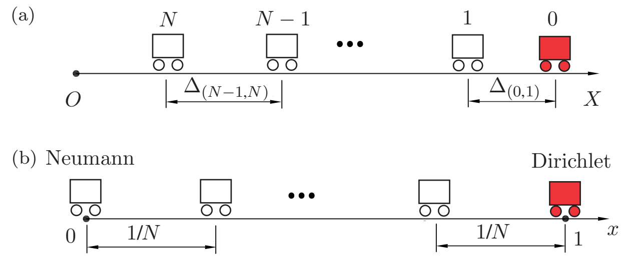

Figure 1. Desired geometry of a platoon with N vehicles and 1 'reference vehicle', moving in 1D Euclidean space. The 'filled' vehicle in the front of the platoon represents the reference vehicle, whose index is '0'. (a) The platoon shown in the original p coordinate. (b) The platoon redrawn in the $\tilde{p}$ coordinate.

formation is specified by the *desired gaps* $\Delta_{(i-1,i)}$ for $i \in \{1, \ldots, N\}$, where $\Delta_{(i-1,i)}$ is the desired value of $p_{i-1}(t) - p_i(t)$.

We consider the following *decentralised* control law used in Barooah et al. (2009), whereby the control action at the i-th vehicle depends on the relative position measurements with its immediate neighbours in the platoon, its own velocity and the desired velocity v^* of the platoon, which is assumed to be known to every vehicle:

$$\begin{aligned} u_i = &-k_i^f(p_i - p_{i-1} + \Delta_{(i-1,i)}) - k_i^b(p_i - p_{i+1} - \Delta_{(i,i+1)}) \\ &- b_i(\dot{p}_i - v^*), \end{aligned} \tag{1}$$

where $i \in \{1, \ldots, N-1\}$, and k_i^f, k_i^b are the front and back position gains and b_i is the velocity gain of the i-th vehicle. For the vehicle with index N, the control law is

$$u_N = -k_N^f(p_N - p_{N-1} + \Delta_{(N-1,N)}) - b_N(\dot{p}_N - v^*), \tag{2}$$

since it does not have a neighbour behind it. Each vehicle i knows the desired gaps $\Delta_{(i-1,i)}$, $\Delta_{(i,i+1)}$, while only vehicle 1 knows the desired trajectory $p_0^*(t)$ of the fictitious reference vehicle. For the ease of analysis, the tracking error $\tilde{p}_i := p_i - p_i^*$ is introduced. The closed-loop dynamics can now be expressed as the following coupled-ODE model (see Barooah et al. (2009)) for more details):

$$\begin{aligned} \ddot{\tilde{p}}_i + b_i\dot{\tilde{p}}_i &= -k_i^f(\tilde{p}_i - \tilde{p}_{i-1}) - k_i^b(\tilde{p}_i - \tilde{p}_{i+1}), \\ &\quad i \in \{1, \ldots, N-1\}, \\ \ddot{\tilde{p}}_N + b_N\dot{\tilde{p}}_N &= -k_N^f(\tilde{p}_N - \tilde{p}_{N-1}). \end{aligned} \tag{3}$$

Using the fact that $\tilde{p}_0(t) \equiv 0$, the coupled-ODEs (3) can be represented in the following state space form:

$$\dot{X} = AX, \tag{4}$$

with state vector $X = [\tilde{p}_1, \dot{\tilde{p}}_1, \ldots, \tilde{p}_N, \dot{\tilde{p}}_N] \in \mathbb{R}^{2N}$. The eigenvalues of A appear in (real or conjugate) pairs, they are denoted by $\varsigma_\ell^\pm$, $\ell \in \{1, \ldots, N\}$, and the least stable eigenvalue (the one closest to the imaginary axis) is denoted by $\varsigma_{\min} := \varsigma_1^+$.

In Barooah et al. (2009), a PDE was derived as an approximation of the coupled-ODE model (3) for large N. We provide a sketch of the deviation of this PDE model, interested readers are referred to Barooah et al. (2009) for the details. We first rewrite the couple-ODE model (3) as

$$\ddot{\tilde{p}}_i + b_i \dot{\tilde{p}}_i = \frac{k_i^f - k_i^b}{N} \frac{(\tilde{p}_{i-1} - \tilde{p}_{i+1})}{2(1/N)} + \frac{k_i^f + k_i^b}{2N^2} \frac{(\tilde{p}_{i-1} - 2\tilde{p}_i + \tilde{p}_{i+1})}{1/N^2}. \tag{5}$$

To facilitate the analysis, we redraw the graph of the 1D platoon shown in Figure 1(a), so that the position of the vehicles in the graph are always located in the interval [0, 1], irrespective of the number of vehicles in the platoon. The i-th agent in the 'original' graph is now drawn at the position $(N-i)/N$ in the new graph Figure 1(b). Figure 1 shows an example. In the limit when $N \to \infty$, Equation (5) can be seen as a finite difference discretisation of the following PDE:

$$\frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} + b(x) \frac{\partial \tilde{p}(x,t)}{\partial t} = \frac{k^f(x) - k^b(x)}{N} \frac{\partial \tilde{p}(x,t)}{\partial x} + \frac{k^f(x) + k^b(x)}{2N^2} \frac{\partial^2 \tilde{p}(x,t)}{\partial x^2}, \tag{6}$$

with boundary conditions

$$\frac{\partial \tilde{p}}{\partial x}(0,t) = 0, \quad \tilde{p}(1,t) = 0, \tag{7}$$

where $k^f(x), k^b(x), b(x) : [0, 1] \to \mathbb{R}_+$ are the continuous approximations of the gains k_i^f, k_i^b, b_i with the following stipulation:

$$k_i^f = k^f(x)|_{x=\frac{N-i}{N}}, \quad k_i^b = k^b(x)|_{x=\frac{N-i}{N}}, \quad b_i = b(x)|_{x=\frac{N-i}{N}}. \tag{8}$$

The PDE model (6)–(7) is an approximation of the coupled-ODE model (3) in the sense that a finite difference discretisation of the PDE yields (3). Existence of solution of the PDE is examined in Barooah, Mehta, and Hespanha (2008).

The situation considered in Barooah et al. (2009) was $b_i = b_0 > 0$ for all i (correspondingly $b(x) = b_0$). To define stability margin of the resulting PDE model, we take the Laplace transform of both sides with respect to the time variable to obtain

$$(s^2 + b_0 s)\eta(x,s) = \mathcal{P}\eta(x,s), \tag{9}$$

where $\eta(x,s) := \mathcal{L}(\tilde{p}(x,t))$ is the Laplace transform of $\tilde{p}(x,t)$ and the operator $\mathcal{P}$ is defined as

$$\mathcal{P} := \frac{k^f(x) - k^b(x)}{N} \frac{\partial}{\partial x} + \frac{k^f(x) + k^b(x)}{2N^2} \frac{\partial^2}{\partial x^2}. \tag{10}$$

Using the method of separation of variables (Haberman 2003; Evans 2010), we assume a solution of the form $\eta(x,s) = \sum_{\ell=1}^{\infty} H(s)\phi_\ell(x)$, where the eigenpairs $(\phi_\ell(x), -\mu_\ell)$ solve the continuous (Sturm–Liouville) eigenvalue problem

$$\mathcal{P}\phi_\ell(x) = -\mu_\ell \phi_\ell(x), \quad \ell = 1, 2, \ldots. \tag{11}$$

From (9)–(11), we have the following characteristic equation for the PDE model

$$s^2 + b_0 s + k_0 \mu_\ell = 0. \tag{12}$$

For each $\ell \in \{1, 2, \ldots\}$, the two roots of the characteristic equations are denoted by $\nu_\ell^{\pm}$. The one that is closer to the imaginary axis is denoted by ν_ℓ^+, and is called the *less stable* eigenvalue between the two. The set $\cup_\ell \nu_\ell^{\pm}$ constitutes the eigenvalues of the PDE (6). The *least stable* eigenvalue among them is $\nu_{\min} := \nu_1^+$.

2.2 *Main results*

We formally define symmetric control, homogeneity and stability margin before stating the main results, i.e. the stability margin approximation errors of the platoon with symmetric and asymmetric control.

Definition 2.1: The control law (1) is symmetric if each agent uses the same front and back position gains: $k_i^f = k_i^b$, for all $i \in \{1, 2, \ldots, N-1\}$, and is called homogeneous if $k_i^f = k_j^f$, $k_i^b = k_j^b$ for every pair (i, j) and $b_i = b_0$ for some b_0 for each i.

Definition 2.2: The stability margin of the coupled-ODE model, denoted by S_o, is the absolute value of the real part of the least stable eigenvalue of A in (4), i.e. $S_o := |\text{Re}(\varsigma_{\min})|$. The stability margin of the PDE model (6)–(7), denoted by S_p, is the absolute value of the real part of the least stable eigenvalue of the PDE, i.e. $S_p := |\text{Re}(\nu_{\min})|$.

We first summarise the results from Barooah et al. (2009) that were derived by analysing the PDE model. Apart from symmetric and homogeneous control, (Barooah et al. 2009) examined the question of optimal design of gain profiles $k^f(x)$, $k^b(x)$ subject to the constraint of *small* asymmetry and inhomogeneity: $|k^f(x) - k_0|/k_0 < \epsilon$, $|k^b(x) - k_0|/k_0 < \epsilon$, where $k_0 > 0$ is the nominally symmetric position gain and ϵ is a small positive number, denoting the amount of asymmetry.[1] We use $S_o^{(0)}, S_o^{(\epsilon)}$ (resp., $S_p^{(0)}, S_p^{(\epsilon)}$) to denote the stability margin for the coupled-ODE (resp., PDE) with symmetric control and the 'optimal' asymmetric control, respectively, which are described next.

Proposition 2.3 (Corollaries 1 and 3 of Barooah et al. 2009)**:** Consider an N-vehicle platoon with closed-loop dynamics (4).

(1) With symmetric control, the stability margin of the PDE model (6)–(7) is $S_p^{(0)} = O(1/N^2)$.
(2) The optimal control gains, in the limit of small ϵ, are given by $k^f(x) = k_0(1+\epsilon)$, $k^b(x) = k_0(1-\epsilon)$, and the resulting stability margin of the PDE model (6)–(7) is $S_p^{(\epsilon)} = O(\epsilon/N)$.

□

Remark 2.4: Proposition 2.3 shows that with symmetric control, the stability margin decays to 0 as $O(1/N^2)$, irrespective of how the control gains k_0 and b_0 are chosen (as long as they are constants independent of N). The reason why we have the $O(1/N^2)$ scaling trend is because with symmetric control the coefficient of the $\partial^2/\partial x^2$ term in the PDE (6) is $O(1/N^2)$ and the coefficient of the $\partial/\partial x$ term is 0. However, any asymmetry between the forward and the backward position gains will lead to non-zero $k^f(x) - k^b(x)$ and a presence of $O(\frac{1}{N})$ term as the coefficient of $\partial/\partial x$. By a judicious choice of asymmetry, there is thus a potential to improve the stability margin from $O(1/N^2)$ to $O(1/N)$. Proposition 2.3 shows that this can indeed be achieved, and provides a control design that leads to the maximal improvement in the stability margin within the prespecified bounds on the control gains. Note that the coupled ODE-model provides no such insight into the effect of asymmetric control gains on the stability margin. Another interesting fact that comes out of the PDE-based design is that heterogeneity is not crucial for the control design. In fact, the control design that leads to the maximal improvement in the stability margin is a homogeneous one. □

The topic of this article is the approximation error in modelling the coupled ODE (3) with the PDE model (6). Though the design in Barooah et al. (2009) was based on the PDE model, and the design was numerically corroborated, the error introduced in the PDE approximation was not analysed. This leaves the question open on how much the results from PDE-based analysis can be trusted. The next theorem, which is the main result of this article, provides an answer to this question.

The proof of the theorem is provided in the subsequent sections. The theorem quantifies the difference $S_o - S_p$ between the stability margin of the coupled-ODE (3) and the PDE (6)–(7), and thereby the error introduced in approximating the stability margin of the coupled-ODE by that of the PDE. We therefore call this difference the *approximation error*.

Theorem 2.5: Consider an N-vehicle platoon with closed-loop dynamics (4). With the asymmetric control gains specified in Proposition 2.3, we have $|S_p^{(\epsilon)} - S_o^{(\epsilon)}| = O(\epsilon/N^2) + O(\epsilon^2) + O(1/N^3)$, where the results hold in the limit $N \to \infty$ and $\epsilon \to 0$. □

The following corollaries are immediate.

Corollary 2.6: With symmetric and homogeneous control, we have $|S_p^{(0)} - S_o^{(0)}| = O(1/N^3)$, where the results hold in the limit $N \to \infty$. □

Corollary 2.7: In both the symmetric and asymmetric cases, we have $(|S_p^{(\cdot)} - S_o^{(\cdot)}|)/(S_o^{(\cdot)}) = O(1/N)$, where the results hold in the limit $N \to \infty$ and $\epsilon \to 0$. □

These results show that the error due to the PDE approximation is negligible as long as the number of vehicles N is large.

3. Stability margins of the coupled-ODE and PDE models

3.1 *Stability margin of the coupled-ODE model*

We first consider the optimal asymmetric control case, where $k_i^f = (1+\epsilon)k_0, k_i^b = (1-\epsilon)k_0, \quad b_i = b_0,$ where $k_0 > 0$, $b_0 > 0$ are the nominally symmetric position and velocity gains, respectively. Note that symmetric control is a special case, obtained by setting $\epsilon = 0$. By defining the vector $\psi := [\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_N]^T \in \mathbb{R}^N$, the closed-loop dynamics of the platoon can now be written compactly from (3) as

$$\ddot{\psi} + B\dot{\psi} = -k_0 L^{(\epsilon)} \psi, \tag{13}$$

where $B = b_0 I$, with I being the $N \times N$ identity matrix, and $L^{(\epsilon)}$ is given by

$$L^{(\epsilon)} = \begin{bmatrix} 2 & -1+\epsilon & & & \\ -1-\epsilon & 2 & -1+\epsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -1-\epsilon & 2 & -1+\epsilon \\ & & & -1-\epsilon & 1+\epsilon \end{bmatrix}. \tag{14}$$

We assume that $\lambda_\ell^{(\epsilon)}$ $(\ell \in \{1, 2, \ldots, N\})$ solves the discrete eigenvalue problem

$$L^{(\epsilon)} v_\ell^{(\epsilon)} = \lambda_\ell^{(\epsilon)} v_\ell^{(\epsilon)}, \tag{15}$$

with an associated eigenvector $v_\ell^{(\epsilon)}$. Substituting (15) into (13) and taking the Laplace transform with respect to the time variable, we obtain the following characteristic equation for the coupled-ODE model:

$$s^2 + b_0 s + k_0 \lambda_\ell^{(\epsilon)} = 0. \tag{16}$$

The two roots are $\varsigma_\ell^{\pm} := \frac{1}{2}(-b_0 \pm \sqrt{b_0^2 - 4k_0\lambda_\ell^{(\epsilon)}})$. If the discriminant is positive, both the eigenvalues are real-valued, with ς_ℓ^{+} being the less stable between the two. The *least stable* eigenvalue is then

$$\varsigma_{\min} = \min_\ell \varsigma_\ell^{+} = \frac{-b_0 + b_0\sqrt{1 - 4k_0\lambda_1^{(\epsilon)}/b_0^2}}{2}, \tag{17}$$

where $\lambda_1^{(\epsilon)}$ is the *principal* (smallest) eigenvalue of $L^{(\epsilon)}$. For small $\lambda_1^{(\epsilon)}$, a Taylor series expansion of the square root term leads to the following expression for the stability margin of the coupled-ODE model (3)

$$S_o^{(\epsilon)} := |Re(\varsigma_{\min})| = \frac{k_0\lambda_1^{(\epsilon)}}{b_0} + O\left(\left(\lambda_1^{(\epsilon)}\right)^2\right), \tag{18}$$

which will be used in the subsequent analysis.

3.2 *Stability margin of the PDE model*

For optimal asymmetric control gains given in Proposition 2.3, we have $k^f(x)+k^b(x)=2k_0$, $k^f(x)-k^b(x)=2\epsilon k_0$, $b(x)=b_0$. Substituting these into the PDE model (6), we have

$$\frac{\partial^2\tilde{p}(x,t)}{\partial t^2} + b_0\frac{\partial\tilde{p}(x,t)}{\partial t} = \epsilon\frac{2k_0}{N}\frac{\partial\tilde{p}(x,t)}{\partial x} + \frac{k_0}{N^2}\frac{\partial^2\tilde{p}(x,t)}{\partial x^2}. \tag{19}$$

Using separation of variables, we assume a solution of the form $\tilde{p}(x,t) = \sum_{\ell=1}^{\infty}\phi_\ell^{(\epsilon)}(x)h_\ell(t)$, we obtain the following:

$$\ddot{h}_\ell(t) + b_0\dot{h}_\ell(t) + \mu_\ell^{(\epsilon)}k_0 = 0, \tag{20}$$

where $\mu_\ell^{(\epsilon)}$ solves the following continuous (Sturm–Liouville) eigenvalue problem:

$$\frac{\mathrm{d}\phi_\ell^{(\epsilon)}(x)}{\mathrm{d}x^2} + \epsilon 2N\frac{\phi_\ell^{(\epsilon)}(x)}{\mathrm{d}x} + \mu_\ell^{(\epsilon)}N^2\phi^{(\epsilon)}(x) = 0, \tag{21}$$

with the following boundary condition, which comes from (7):

$$\frac{\mathrm{d}\phi_\ell^{(\epsilon)}(0)}{\mathrm{d}x} = 0, \quad \phi_\ell^{(\epsilon)}(1) = 0. \tag{22}$$

Taking a Laplace transform of (20), we obtain the characteristic equation for the PDE model:

$$s^2 + b_0 s + k_0\mu_\ell^{(\epsilon)} = 0, \quad \ell = 1,2,\ldots. \tag{23}$$

Following the same analysis as the coupled-ODE model, we obtain the following expression for the stability margin of the PDE model (6)–(7):

$$S_p^{(\epsilon)} = \frac{k_0\mu_1^{(\epsilon)}}{b_0} + O\left(\left(\mu_1^{(\epsilon)}\right)^2\right). \tag{24}$$

Comparing (18) with (24), we see that the stability margin approximation error is given by

$$S_p^{(\epsilon)} - S_o^{(\epsilon)} = \frac{k_0}{b_0}\left(\lambda_1^{(\epsilon)} - \mu_1^{(\epsilon)}\right) + O\left(\left(\lambda_1^{(\epsilon)}\right)^2\right) + O\left(\mu_1^{(\epsilon)}\right)^2. \tag{25}$$

Estimates of the discrete and continuous eigenvalues $\lambda_1^{(\epsilon)}$ and $\mu_1^{(\epsilon)}$ are obtained in the next section.

4. Stability margin approximation errors

We will now develop the stability margin approximation errors in terms of N and ϵ based on the expression in (25) derived in the previous section. With symmetric control, explicit formulae can be found for the discrete and continuous eigenvalues $\lambda_1^{(0)}$ and $\mu_1^{(0)}$. However, for asymmetric control, in general there are no explicit solutions. We will use perturbation methods to derive accurate bounds for $\lambda_1^{(\epsilon)}$ and $\mu_1^{(\epsilon)}$ under the assumption that ϵ is sufficiently small.

4.1 *Stability margin approximation error with symmetric control*

The proof of the main theorem requires certain intermediate result that leads to Corollary 2.6 first. We present the result here. In the case of symmetric control, $\epsilon=0$, and so we have from (14):

$$L^{(0)} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}. \tag{26}$$

The principal eigenvalue of $L^{(0)}$ and the corresponding eigenvector are given by Yueh (2005)

$$\lambda_1^{(0)} = 4\sin^2\left(\frac{\pi}{2(2N+1)}\right),$$
$$v_1^{(0)}(k) = \sin\left(\frac{k\pi}{2N+1}\right), k\in\{1,\ldots,N\}. \tag{27}$$

By a Taylor series expansion, we have

$$\lambda_1^{(0)} = \frac{\pi^2}{4N^2} - \frac{\pi^2}{4N^3} + O\left(\frac{1}{N^4}\right). \tag{28}$$

For the PDE model, to get an explicit expression for $\mu^{(0)}$, we consider the following continuous eigenvalue problem with the same boundary condition as (22), which is obtained by setting $\epsilon=0$ in (21):

$$\frac{d^2\phi_\ell^{(0)}(x)}{dx^2} + \mu_\ell^{(0)}N^2\phi_\ell^{(0)}(x) = 0. \tag{29}$$

Following straightforward algebra (Haberman 2003), the smallest eigenvalue and its corresponding eigenfunction are given by

$$\mu_1^{(0)} = \frac{\pi^2}{4N^2}, \quad \phi_1^{(0)}(x) = \cos\left(\frac{\pi}{2}x\right). \tag{30}$$

Upon obtaining the above results for $\lambda_1^{(0)}$ and $\mu_1^{(0)}$, the proof of Corollary 2.6 is straightforward.

Proof 4.1 (Proof of Corollary 2.6)**:** Using (28) and (30) in (25), we get

$$S_o^{(0)} - S_p^{(0)} = \frac{k_0}{b_0}\left(-\frac{\pi^2}{4N^3}\right) + O\left(\frac{1}{N^4}\right) = O\left(\frac{1}{N^3}\right),$$

which proves the result. □

4.2 *Stability margin approximation error with asymmetric control*

When the control is asymmetric ($\epsilon \neq 0$), it is easy to see that $L^{(\epsilon)}$ can be expressed as $L^{(\epsilon)} = L^{(0)} + \epsilon\tilde{L}$, where

$$\tilde{L} = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 1 \end{bmatrix}. \quad (31)$$

From Ngo (2005), we have that for $|\epsilon| \ll 1$, the perturbed eigenvalue $\lambda_1^{(\epsilon)}$ of $L^{(\epsilon)}$ can be written as

$$\lambda_1^{(\epsilon)} = \lambda_1^{(0)} + \epsilon\frac{\left\langle v_1^{(0)}, \tilde{L}v_1^{(0)}\right\rangle}{\left\langle v_1^{(0)}, v_1^{(0)}\right\rangle} + O(\epsilon^2), \quad (32)$$

where $\langle ., .\rangle$ denotes the standard inner product and $\lambda_1^{(0)}$ is the principal eigenvalue of $L^{(0)}$ and $v_1^{(0)}$ is its associated eigenvector given in (27). It is straightforward to show that

$$\tilde{L}v_1^{(0)} = 2\sin\frac{\pi}{2N+1}\left[\cos\frac{\pi}{2N+1}, \ldots, \cos\frac{N\pi}{2N+1}\right].$$

We now have

$$\begin{aligned}\left\langle v_1^{(0)}, \tilde{L}v_1^{(0)}\right\rangle &= \sin\frac{\pi}{2N+1}\sum_{k=1}^{N}\sin\frac{2k\pi}{2N+1} \\ &= \frac{1}{2} + \frac{1}{2}\cos\frac{\pi}{2N+1},\end{aligned} \quad (33)$$

$$\begin{aligned}\left\langle v_1^{(0)}, v_1^{(0)}\right\rangle &= \sum_{k=1}^{N}\sin^2\frac{k\pi}{2N+1} \\ &= \frac{N}{2} - \frac{1}{2}\sum_{k=1}^{N}\cos\frac{2k\pi}{2N+1} = \frac{2N+1}{4},\end{aligned} \quad (34)$$

where the last equalities in (33) and (34) follow from the following facts (Lin 2001):

$$\sum_{i=1}^{N}\sin(ix) = \frac{\cos\frac{x}{2} - \cos(N+\frac{1}{2})x}{2\sin\frac{x}{2}},$$

$$\sum_{i=1}^{N}\cos(ix) = \frac{\sin(N+\frac{1}{2})x - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Combining the above results, we have the following expression for $\lambda_1^{(\epsilon)}$:

$$\begin{aligned}\lambda_1^{(\epsilon)} &= 4\sin^2\frac{\pi}{2(2N+1)} + \epsilon\frac{2(1+\cos\frac{\pi}{2N+1})}{2N+1} \\ &= \frac{\pi^2}{4N^2} - \frac{\pi^2}{4N^3} + \epsilon\frac{2}{N} - \epsilon\frac{1}{N^2} + O(\epsilon^2) + O\left(\frac{1}{N^4}\right).\end{aligned} \quad (35)$$

For the PDE model, the analysis of the continuous eigenvalue problem (21) also proceeds by a perturbation method when $\epsilon \neq 0$. For vanishingly small ϵ, we assume the smallest eigenvalue and its eigenfunction of the form (Haberman 2003; Chapter 9)

$$\begin{aligned}\mu_1^{(\epsilon)} &= \mu_1^{(0)} + \epsilon\tilde{\mu}_1 + O(\epsilon^2), \\ \phi_1^{(\epsilon)}(x) &= \phi_1^{(0)}(x) + \epsilon\tilde{\phi}_1(x) + O(\epsilon^2).\end{aligned} \quad (36)$$

Substituting the above result into (21), and doing an $O(1)$ balance, we have

$$\frac{d^2\phi_1^{(0)}(x)}{dx^2} + \mu_1^{(0)}N^2\phi_1^{(0)}(x) = 0. \quad (37)$$

This is the continuous eigenvalue problem for symmetric control case ($\epsilon = 0$) whose solution is given in (30). Doing an $O(\epsilon)$ balance, we have the following:

$$\begin{aligned}\frac{d^2\tilde{\phi}_1(x)}{dx^2} + \mu_1^{(0)}N^2\tilde{\phi}_1(x) &= -2N\frac{d\phi_1^{(0)}(x)}{dx} \\ &\quad - \tilde{\mu}_1N^2\phi_1^{(0)}(x) =: R.\end{aligned} \quad (38)$$

From the Fredholm alternative (refer to Haberman (2003; Chapter 9)), for a solution $\tilde{\phi}_1(x)$ to exist, R must be orthogonal to $\phi_1^{(0)}$, so we have

$$\int_0^1 \phi_1^{(0)}(x)\left(2N\frac{d\phi_1^{(0)}(x)}{dx} + \tilde{\mu}_1N^2\phi_1^{(0)}(x)\right)dx = 0, \quad (39)$$

which yields $\tilde{\mu}_1 = \frac{2}{N}$, and therefore

$$\mu_1^{(\epsilon)} = \frac{\pi^2}{4N^2} + \epsilon\frac{2}{N} + O(\epsilon^2). \quad (40)$$

We are now ready to present the proofs of Theorem 2.5 and Corollary 2.7.

Proof 4.2 (Proof of Theorem 2.5)**:** Using (35) and (40) in (25), we obtain

$$S_o^{(\epsilon)} - S_p^{(\epsilon)} = -\frac{k_0}{b_0}\frac{\epsilon}{N^2} + O\left(\frac{1}{N^3}\right) + O\left(\frac{\epsilon^2}{N^2}\right) = O\left(\frac{\epsilon}{N^2}\right),$$

where the last equality follows from the assumptions that N is large and ϵ is small. □

Proof 4.3 (Proof of Corollary 2.7)**:** By substituting (28) and (35) into (18) and ignoring the higher order terms, we have that for symmetric and asymmetric controls, the stability margins of the coupled-ODE

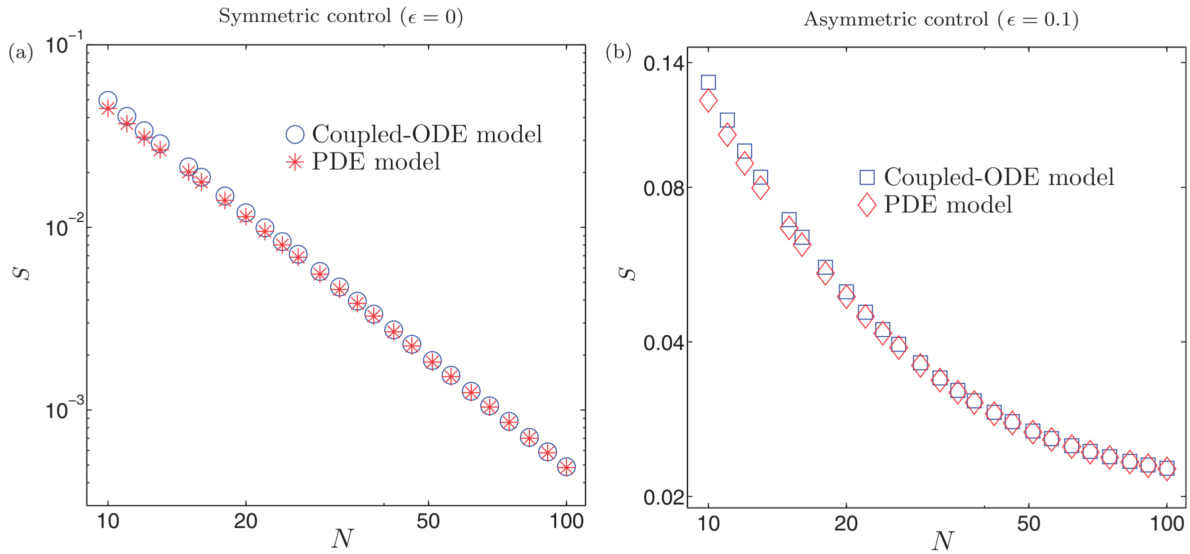

Figure 2. Stability margin comparisons between the coupled-ODE and PDE models with symmetric and asymmetric control. In the above figures, S denotes stability margin and N denotes the number of vehicles in the platoon.

model are, respectively, $S_o^{(0)} = O(1/N^2)$ and $S_o^{(\epsilon)} = O(\epsilon/N)$. The proof follows immediately by using Corollary 2.6 and Theorem 2.5. □

4.3 *Numerical comparison*

Figure 2 depicts the comparison between the stability margins of the couple-ODE model (3) and the PDE model (6). The stability margin of the coupled-ODE model is obtained by evaluating the eigenvalues of the closed-loop state matrix A. For the PDE model, we use the Galerkin projection method to compute the stability margin (Canuto, Hussaini, Quarteroni, and Zang 1983). Figure 2(a) shows the stability margin comparison result with symmetric control where the control gains for the coupled-ODE model are specified as $k_i^f = k_i^b = k_0 = 1$ and $b_i = b_0 = 0.5$. The corresponding control gains for the PDE model are $k^f(x) = k^b(x) = k_0 = 1$ (which lead to $k^f(x) - k^b(x) = 0$, $k^f(x) + k^b(x) = 2k_0 = 2$) and $b(x) = b_0 = 0.5$; Figure 2(b) shows the stability margin comparison result for an asymmetric control case where the amount of asymmetry is given by $\epsilon = 0.1$. The control gains for the coupled-ODE are $k_i^f = (1+\epsilon)k_0 = 1.1, k_i^b = (1-\epsilon)k_0 = 0.9$ and $b_i = b_0 = 0.5$. For the corresponding PDE model, the control gains become $k^f(x) - k^b(x) = 2\epsilon k_0 = 0.2$, $k^f(x) + k^b(x) = 2k_0 = 2$ and $b(x) = b_0 = 0.5$. We can see from Figure 2 that for both the symmetric and asymmetric control cases, the stability margin of the PDE model is an accurate approximation of the stability margin of the coupled-ODE model when N is large, which verifies the results of this article.

5. Conclusion

We studied the error introduced in modelling the closed-loop dynamics of a large vehicular platoon by using a PDE. The value of the PDE approximation is that it provides powerful insights into the effect of control architecture on the stability margin, in particular, the beneficial effect of front-back asymmetry (refer to Remark 2.4). The insight from the PDE was used in prior works by Barooah et al. (2009) and Hao et al. (2011) to design an asymmetric control architecture that improves the stability margin of the closed loop to $O(1/N)$ from the much poorer scaling law of $O(1/N^2)$ that results from symmetric control.

The aforementioned papers, however, provided no rigorous analysis on how well the PDE model approximates the coupled ODE model. Instead, a design derived from the PDE model was directly implemented on the coupled-ODE model and the resulting improvement was numerically verified. The contribution of this article is to provide a rigorous analysis of the approximation error. The main results of this article (Theorem 2.5 and its corollaries) show that the error due to the continuum approximation is negligible if N is large.

The results of this article, and in fact those in Barooah et al. (2009) and Hao et al. (2011), hold only for the case of small asymmetry, i.e. when ϵ is vanishingly small. This is due to the use of perturbation-based analysis to obtain the results for the asymmetric case ($\epsilon \neq 0$). Analysis of the approximation error between the two models for non-vanishing asymmetry and arbitrary choice of control gains is the subject of ongoing work. Another topic of interest is to study the eigenvalue approximation error between general Sturm–Liouville operator and its discretisation.

The analysis in this article is limited to a specific boundary condition (Dirichlet–Neumann) that corresponds to the scenario when there is a fictitious lead vehicle in front of the platoon. Similar results hold for the Dirichlet–Dirichlet boundary condition that corresponds to the scenario when there are fictitious lead and follow vehicles on both ends of the platoon. The analysis can be carried out in a manner exactly analogous to that in this article. Numerical verification of this statement is provided in Barooah et al. (2009; Figure 6).

Whereas the PDE model in this article corresponds to a platoon where each vehicle uses a relative position and an absolute velocity feedback, a slightly different PDE is obtained if one considers relative position and relative velocity feedback (Hao and Barooah 2010). The approach that led to the control design based on the PDE (6) in Barooah et al. (2009) also led to an improved control design in Hao and Barooah (2010). While the PDE (6) is for a one-dimensional platoon, a corresponding PDE was derived in Hao et al. (2011) for a *d*-dimensional vehicular formation. Again, a PDE-based design was carried out that led to an improved stability margin over the symmetric design. Thus, the PDE-based approach for a distributed control design is useful to a range of distributed control problems involving double-integrator agents.

Besides stability margin, the asymmetric designs that were arrived by the PDE-based analysis were also found to improve the closed-loop's robustness to external disturbances (Barooah et al. 2009; Hao and Barooah 2010; Lin, Fardad, and Jovanovic 2012). Thus, the PDE model is beneficial in studying multiple aspects of the distributed control design (both stability and robustness). In this article we limited ourselves to the question of PDE approximation error for the stability margin. A more comprehensive investigation that includes the robustness question is a subject of future work.

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Note

1. The case considered in Barooah et al. (2009) was, in fact, $|k^f(x) - k_0| < \epsilon$, $|k^b(x) - k_0| < \epsilon$. It is straightforward, however, to re-derive the results if the constraints are changed to the form used here: $|k^f(x) - k_0|/k_0 < \epsilon$, $|k^b(x) - k_0|/k_0 < \epsilon$. In this paper we consider the latter case since it makes the analysis cleaner without changing the results of Barooah et al. (2009) significantly.

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