2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Asymmetric control achieves size-independent stability margin in 1-D flocks

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***Abstract*— We consider the stability margin of a large 1-D flock of double-integrator agents with distributed control, in which the control at each agent depends on the relative information from its nearest neighbors. The stability margin is measured by the real part of the least stable eigenvalue of the closed-loop state matrix, which quantifies the rate of decay of initial errors. In [1], it was shown that with symmetric control, in which two neighbors put equal weight on information received from each other, the stability margin of the flock decays to 0 as $O(1/N^2)$, where N is the number of agents. Moreover, a perturbation analysis was used to show that with vanishingly small amount of asymmetry in the control gains, the stability margin can be improved to $O(1/N)$. In this paper, we show that, in fact, with asymmetric control the stability margin of the closed-loop can be bounded away from zero uniformly in N. Asymmetry in control gains thus makes the control architecture highly scalable. We establish the results through distinct routes, using state-space analysis and also using a partial differential equation (PDE) approximation. Numerical verifications are also provided to corroborate our analysis.**

I. Introduction

The problem of distributed control of multiple agents is relevant to many applications such as automated highway system, collective behavior of bird flocks and animal swarms, and formation flying of unmanned aerial and ground vehicles for surveillance, reconnaissance and rescue, etc. [2]–[6]. A classical problem in this area is the distributed formation control of a 1-D flock of agents, in which each agent is modeled as a double integrator. The control action at each agent is based on the information from its two nearest neighbors (one on either side). The control objective is to make the flock track a desired trajectory while maintaining a rigid formation geometry. The desired trajectory of the entire formation is given in terms of a fictitious reference agent, and the desired formation geometry is specified in terms of constant inter-agent spacings.

A typical issue faced in this problem is that the performance of the closed-loop degrades as the number of agents increases. Several recent papers have studied the scaling of performance of formations of double-integrator agents as a function of the number of agents. In particular, [1], [7] have studied the scaling of the stability margin, while [8]–[11] have examined the sensitivity to external disturbances. However, most of the work impose the condition that the information graph is undirected (i.e., symmetric), which means that between two agents i and j that exchange information, the weight placed by i on the information received from j is the same as the weight placed by j on that received from i. In a previous paper [1], it was shown that with symmetric control, the stability margin of the 1-D flock, which is measured by the real part of the least stable eigenvalue of the closed-loop, decays to 0 as $O(1/N^2)$, where N is the number of agents.

In this paper, we study the stability margin of a large 1-D flock of double-integrator agents whose information graph is directed or asymmetric. Little work has been done on coordination of double integrator agents with directed information graphs, with [1], [11] being exceptions. It was shown in [1] that with vanishingly small asymmetry in the control gains, the stability margin can be improved to $O(1/N)$. Similar conclusions are also obtained for a vehicular formation with a D-dimensional lattice as its information graph [7]. The analyses in [1], [7] were based on a partial differential equation (PDE) approximation of the closed-loop dynamics and a perturbation method; the latter limited the results to only vanishingly small asymmetry. The reference [11] studied the effect of asymmetry in control on the flock's sensitivity to disturbances, but not its stability margin.

In this paper, we show that with a fixed amount of asymmetry in the control gains, the stability margin of the flock can be uniformly bounded away from 0 (independent of N). This stronger result - compared to those in [1], [7] – is obtained by avoiding the perturbation analysis of the aforementioned papers. We provide two alternate proofs of the result. One line of analysis proceeds with the PDE-approximation of the coupled-ODE model that was used in [1], [7]. Techniques from Strum-Liouville theory are used to derive a closed-form expression for the lower bound, which is then used to establish that the lower bound is independent of N. The second line of analysis deals with the coupled-ODE model directly. The advantage of the PDE-based analysis is that it provides powerful insights on the benefits of asymmetric control on the performance of the system.

We also show that the smallest eigenvalue of the *directed grounded Laplacian* of the information graph plays a pivotal role in determining the stability margin of the system. Although our study is focused on agents with double integrator dynamics, eigenvalues of digraphs are also important in the study of convergence rate of distributed consensus, which is essentially coordination of vehicles with single-integrator dynamics. Even in the consensus literature, study of the graph Laplacian spectra for directed graphs has been rather limited [12]. In this paper we provide a formula for the smallest eigenvalues of the directed grounded Laplacian matrix for a 1-D lattice as a function of N.

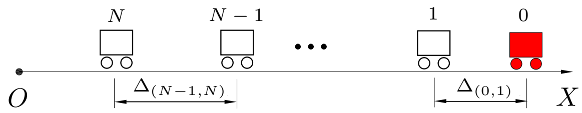

Fig. 1. Desired geometry of a flock with N agent and 1 "reference agent", which are moving in 1D Euclidean space. The filled agent in the front of the flock represents the reference agent, it is denoted by "0".

II. Problem statement and main results

A. Problem statement

We consider the formation control of N identical agents which are moving in 1-D Euclidean space, as shown in Figure 1. The position of the i-th agent is denoted by $p_i \in \mathbb{R}$ and the dynamics of each agent are modeled as a double integrator:

$$\ddot{p}_i = u_i, \quad i \in \{1, 2, \cdots, N\}, \tag{1}$$

where $u_i \in \mathbb{R}$ is the control input, which is the acceleration or deceleration command.

The control objective is that the flock maintains a desired formation geometry while following a constant-velocity type desired trajectory. The desired geometry of the formation is specified by the *desired gaps* $\Delta_{(i-1,i)}$ for $i \in \{1, \cdots, N\}$, where $\Delta_{(i-1,i)}$ is the desired value of $p_{i-1}(t) - p_i(t)$. The desired inter-agent gaps $\Delta_{(i-1,i)}$'s are positive constants and they have to be specified in a mutually consistent fashion, i.e. $\Delta_{(i,k)} = \Delta_{(i,j)} + \Delta_{(j,k)}$ for every triple (i, j, k) where $i \le j \le k$. The desired trajectory of the flock is provided in terms of a *fictitious* reference agent with index "0", whose trajectory is given by $p_0^*(t) = v^* t + c_0$ for some constants v^*, c_0. The desired trajectory of the i-th agent, $p_i^*(t)$, is given by $p_i^*(t) = p_0^*(t) - \Delta_{(0,i)} = p_0^*(t) - \sum_{j=1}^{i} \Delta_{(j-1,j)}$.

We consider the following *decentralized* control law used in [1], whereby the control action at the i-th agent depends on the relative position measurements with its nearest neighbors in the flock (one on either side), its own velocity, and the desired velocity v^* of the flock:

$$\begin{aligned} u_i = &- k_i^f (p_i - p_{i-1} + \Delta_{(i-1,i)}) - k_i^b (p_i - p_{i+1} - \Delta_{(i,i+1)}) \\ &- b_i (\dot{p}_i - v^*), \end{aligned} \tag{2}$$

where $i \in \{1, \cdots, N-1\}$, k_i^f, k_i^b are the front and back position gains and b_i is the velocity gain of the i-th agent. For the agent with index N, the control law is given by:

$$u_N = -k_N^f (p_N - p_{N-1} + \Delta_{(N-1,N)}) - b_N (\dot{p}_N - v^*), \tag{3}$$

since it does not have a neighbor behind it. We assume each agent i knows the desired gaps $\Delta_{(i-1,i)}$, $\Delta_{(i,i+1)}$. To facilitate analysis, we define the tracking error:

$$\tilde{p}_i := p_i - p_i^* \qquad \Rightarrow \qquad \dot{\tilde{p}}_i = \dot{p}_i - \dot{p}_i^*. \tag{4}$$

The closed-loop dynamics can now be expressed as the following coupled-ODE model

$$\begin{aligned} \ddot{\tilde{p}}_i &= -k_i^f (\tilde{p}_i - \tilde{p}_{i-1}) - k_i^b (\tilde{p}_i - \tilde{p}_{i+1}) - b_i \dot{\tilde{p}}_i, \\ \ddot{\tilde{p}}_N &= -k_N^f (\tilde{p}_N - \tilde{p}_{N-1}) - b_N \dot{\tilde{p}}_N. \end{aligned} \tag{5}$$

where $i \in \{1, \cdots, N-1\}$, which can be represented in the following state-space form:

$$\dot{x} = Ax, \tag{6}$$

where $x := [\tilde{p}_1, \dot{\tilde{p}}_1, \cdots, \tilde{p}_N, \dot{\tilde{p}}_N] \in \mathbb{R}^{2N}$ is the state vector.

In [1], a PDE was derived as an approximation of the coupled-ODE model (5) for large N. The PDE governed the evolution of $\tilde{p}(x,t) : [0,1] \times \mathbb{R}_+ \rightarrow \mathbb{R}$, which is a spatially continuous counterpart of the functions $\tilde{p}_i(t)$, $i \in \{1, \cdots, N\}$, with the stipulation that $\tilde{p}_i(t) = \tilde{p}(x,t)|_{x=(N-i)/N}$. The PDE model is given by

$$\begin{aligned} \frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} + b(x) \frac{\partial \tilde{p}(x,t)}{\partial t} = &\frac{k^f(x) - k^b(x)}{N} \frac{\partial \tilde{p}(x,t)}{\partial x} \\ &+ \frac{k^f(x) + k^b(x)}{2N^2} \frac{\partial^2 \tilde{p}(x,t)}{\partial x^2}, \end{aligned} \tag{7}$$

with mixed Dirichlet-Neumann boundary condition

$$\frac{\partial \tilde{p}}{\partial x}(0,t) = 0, \qquad \tilde{p}(1,t) = 0, \tag{8}$$

where $k^f(x), k^b(x), b(x) : [0,1] \rightarrow \mathbb{R}_+$ are the continuous approximations of the gains k_i^f, k_i^b, b_i with the stipulation $k_i^f = k^f(x)|_{x=(N-i)/N}$, $k_i^b = k^b(x)|_{x=(N-i)/N}$, $b_i = b(x)|_{x=(N-i)/N}$.

We refer the reader to [1] for the details of the derivation of the PDE.

B. Main results

We first formally define symmetric control and stability margin before stating the main results.

Definition 1: The control law (2) is *symmetric* if each agent uses the same front and back position gains: $k_i^f = k_i^b$, for all $i \in \{1, 2, \cdots, N-1\}$, and is called *homogeneous* if $k_i^f = k_j^f$, $k_i^b = k_j^b$ and $b_i = b_j$ for every pair (i, j). □

It was shown in [1] that the stability margin can be improved by a large amount by introducing front-back asymmetry in the position feedback gains. Moreover, heterogeneity was found to have little effect on the stability margin [13]. Therefore, we consider the following asymmetric and homogeneous control gains:

$$k_i^f = (1+\epsilon) k_0, \qquad k_i^b = (1-\epsilon) k_0, \qquad b_i = b_0, \tag{9}$$

where $k_0 > 0, b_0 > 0$ are the nominal position and velocity gains respectively, and $\epsilon \in [0, 1)$ denotes the amount of asymmetry. Note that $\epsilon = 0$ corresponds to the symmetric control case. With the control gains given in (9), it's straightforward to see that the state matrix A can be expressed in the following form,

$$A = I_N \otimes A_1 + L_g \otimes A_2, \tag{10}$$

where I_N is the $N \times N$ identity matrix and $\otimes$ denotes the Kronecker product. The matrices A_1, A_2 are defined as below

$$A_1 := \begin{bmatrix} 0 & 1 \\ 0 & -b_0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 0 \\ -k_0 & 0 \end{bmatrix}, \tag{11}$$

and L_g is the *directed grounded Laplacian* of the flock (see Section III):

$$L_g := \begin{bmatrix} 2 & -1+\epsilon & & & \\ -1-\epsilon & 2 & -1+\epsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -1-\epsilon & 2 & -1+\epsilon \\ & & & -1-\epsilon & 1+\epsilon \end{bmatrix}. \quad (12)$$

For the PDE model, the corresponding control gains are $k^f(x) = k_0(1+\epsilon)$, $k^b(x) = k_0(1-\epsilon)$ and $b(x) = b_0$, and the PDE model is simplified to

$$\frac{\partial^2 \tilde{p}(x,t)}{\partial t^2} + b_0 \frac{\partial \tilde{p}(x,t)}{\partial t} = \frac{2\epsilon k_0}{N}\frac{\partial \tilde{p}(x,t)}{\partial x} + \frac{k_0}{N^2}\frac{\partial^2 \tilde{p}(x,t)}{\partial x^2}, \quad (13)$$

To define stability margin of the resulting PDE model (13), we take Laplace transform of both sides with respect to the time variable t and use the method of separation of variables, we have the following characteristic equation for the PDE model (refer to Section IV for more details)

$$s^2 + b_0 s + k_0 \lambda_\ell = 0, \quad \ell \in \{1, 2, \cdots\}, \quad (14)$$

where the eigenpairs $(\lambda_\ell, \phi_\ell(x))$ solve the following boundary value problem

$$\frac{d^2\phi_\ell(x)}{dx^2} + 2\epsilon N \frac{d\phi_\ell(x)}{dx} + \lambda_\ell N^2 \phi_\ell(x) = 0,$$
$$\frac{d\phi_\ell}{dx}(0) = 0, \quad \phi_\ell(1) = 0. \quad (15)$$

For each $\ell \in \{1, 2, \cdots\}$, the two roots of the characteristic equation (14) are denoted by $s_\ell^\pm$. The one that is closer to the imaginary axis is denoted by s_ℓ^+, and is called the *less stable* eigenvalue between the two. The set $\cup_\ell s_\ell^\pm$ constitute the eigenvalues of the PDE (13). The *least stable* eigenvalue among them is denoted by $s_{\min}$.

Definition 2: The *stability margin* of the coupled-ODE model (5), denoted by S_o, is defined as the absolute value of the real part of the least stable eigenvalue of A. The *stability margin* of the PDE model (13) with boundary condition (8), denoted by S_p, is defined as the absolute value of the real part of the least stable eigenvalue of the PDE. □

The following proposition summaries the results in [1].

Proposition 1: Consider an N-agent flock with PDE model (7) and boundary condition (8).

1) [Corollary 1 of [1]] With symmetric control ($\epsilon = 0$), the stability margin of the platoon is $S_p = O(1/N^2)$.
2) [Corollary 3 of [1]] With the asymmetric control gains $k^f(x) = k_0(1+\epsilon)$, $k^b(x) = k_0(1-\epsilon)$ and $b(x) = b_0$, the stability margin of the platoon is $S_p = O(\frac{\epsilon}{N})$. [1]

Statements 1) and 2) hold in the limit $\epsilon \to 0$ and large N. □

Proposition 1 shows that with symmetric control, the stability margin decays to 0 as $O(1/N^2)$, irrespective of how the control gains k_0 and b_0 are chosen (as long as they are constants independent of N). The reason why we have the $O(1/N^2)$ scaling trend is because that with symmetric control the coefficient of the $\frac{\partial^2}{\partial x^2}$ term in the PDE (7) is $O(\frac{1}{N^2})$ and the coefficient of the $\frac{\partial}{\partial x}$ term is 0. However, any asymmetry between the forward and the backward position gains will lead to non-zero $k^f(x) - k^b(x)$ and a presence of $O(\frac{1}{N})$ term as the coefficient of $\frac{\partial}{\partial x}$. By a judicious choice of asymmetry, there is thus a potential to improve the stability margin from $O(\frac{1}{N^2})$ to $O(\frac{1}{N})$. Proposition 1 shows that this can indeed be achieved in the limit of $\epsilon \to 0$. Note that the coupled ODE-model provides no such insight into the effect of asymmetric control gains on the stability margin.

In this paper, we eliminate the restriction that ϵ being vanishingly small and establish the results for arbitrary but fixed ϵ. The following theorems are the main results of this paper, whose proof and numerical corroboration are given in Section III and Section IV respectively. The first theorem is on the stability margin of the coupled-ODE model, and the second is on that of the PDE model.

Theorem 1: With the control gains given in (9) and for any fixed $\epsilon \in (0,\ 1)$, the stability margin S_o of the coupled-ODE model (5) is bounded from below, uniformly in N. Specifically,

$$S_o \geq \frac{\Re\Big(b_0 - \sqrt{b_0^2 - 8k_0(1-\sqrt{1-\epsilon^2})}\Big)}{2}, \quad (16)$$

where $\Re(.)$ denotes the real part. □

The stability margin is $O(1)$ irrespective of whether the expression inside $\Re(\cdot)$ in (16) is real or complex. If complex, the stability margin is simply $b_0/2$. If real, it is straightforward to see that the real part is positive and independent of N, since b_0, k_0, ϵ are constants that do not change with N.

Theorem 2: Consider a flock with PDE model (13) and boundary condition (8). For any fixed $\epsilon \in (0,\ 1)$, the stability margin S_p is bounded from below, uniformly in N. Specifically,

$$S_p \geq \frac{\Re\Big(b_0 - \sqrt{b_0^2 - 4k_0\epsilon^2}\Big)}{2}. \quad □$$

Remark 1: Comparing the results above to the conclusions of [1] that are summarized in Proposition 1, we observe that even with an arbitrarily (but fixed and non-vanishing) amount of asymmetry, the stability margin of the system can be bounded away from zero *uniformly in* N. This asymmetric design therefore makes the resulting control law highly scalable; it eliminates the degradation of closed-loop performance with increasing N. We note that although the control law is the same as that analyzed in [1], the stronger conclusion we obtained - compared to that in [1] - is due to the fact that our analysis does not rely on a perturbation-based technique that was used [1], which limited the analysis in [1] to vanishingly small ϵ. □

[1]The case considered in [1] was that $|k^f(x) - k_0| < \epsilon$, $|k^b(x) - k_0| < \epsilon$. It is straightforward, however, to re-derive the results if the constraints on the gains are changed to the form used here: $|k^f(x) - k_0|/k_0 < \epsilon$, $|k^b(x) - k_0|/k_0 < \epsilon$. In this paper we consider the latter case since it makes the analysis cleaner without changing the results of [1] significantly.

III. Stability margin of the coupled-ODE model of flock dynamics

In this section, we provide a proof of Theorem 1. The analysis of the eigenvalues of the state matrix A relies on the spectrum of the *directed grounded Laplacian* of the flock.

To precisely define the *directed grounded Laplacian* L_g of the flock, recall that the *Laplacian* matrix of a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with n nodes is defined as

$$[L_{N\times N}]_{ij} = \begin{cases} -w(i,j) & i \neq j, (i,j) \in \mathbf{E}, \\ \sum_{k=1}^{N} w(i,k) & i = j, (i,k) \in \mathbf{E}, \\ 0 & \text{otherwise}, \end{cases} \tag{17}$$

where $w(i,j)$ is the weight assigned to the directed edge (i,j). The *directed grounded Laplacian* L_g of $\mathbf{G}$ with respect to a set of grounded nodes $\mathbf{V}_g \subset \mathbf{V}$ is the submatrix of L obtained by removing from L those rows and columns corresponding to the grounded nodes $\mathbf{V}_g$ in $\mathbf{V}$, where $\mathbf{V}_g$ here is the node corresponding to the reference agent. The directed grounded Laplacian L_g of the 1-D flock is given in (12).

We now present a formula for the stability margin of the flock in terms of the smallest eigenvalue of its directed grounded Laplacian.

Lemma 1: With the control gains given in (9) and $0 < \epsilon < 1$, the stability margin of the flock S_o with coupled-ODE model (5) is given by

$$S_o = \begin{cases} \frac{b_0}{2}, & \text{if } \lambda_1 \geq 4k_0/b_0^2, \\ \frac{b_0 - \sqrt{b_0^2 - 4k_0\lambda_1}}{2}, & \text{otherwise.} \end{cases} \tag{18}$$

where λ_1 is the smallest eigenvalue of the directed grounded Laplacian L_g. □

Proof of Lemma 1. Our proof follows a similar line of attack as of [11, Theorem 4.2]. From Schur's triangularization theorem, there exists an unitary matrix U such that

$$U^{-1} L_g U = L_u,$$

where L_u is an upper-triangular matrix, whose diagonal entries are the eigenvalues of L_g. We now do a similarity transformation on matrix A.

$$\begin{aligned} \bar{A} :=& (U^{-1} \otimes I_2) A (U \otimes I_2) \\ =& (U^{-1} \otimes I_2)(I_N \otimes A_1 + L_g \otimes A_2)(U \otimes I_2) \\ =& I_N \otimes A_1 + L_u \otimes A_2. \end{aligned}$$

The above is a block upper-triangular matrix, and the block on each diagonal is $A_1 + \lambda_\ell A_2$, where $\lambda_\ell \in \sigma(L_g)$, where $\sigma(\cdot)$ denotes the spectrum (the set of eigenvalues). Since similarity preserves eigenvalues, and the eigenvalues of a block upper-triangular matrix are the union of eigenvalues of each block on the diagonal, we have

$$\begin{aligned} \sigma(A) = \sigma(\bar{A}) &= \bigcup_{\lambda_\ell \in \sigma(L_g)} \{\sigma(A_1 + \lambda_\ell A_2)\} \\ &= \bigcup_{\lambda_\ell \in \sigma(L_g)} \left\{ \sigma \begin{bmatrix} 0 & 1 \\ -k_0\lambda_\ell & -b_0 \end{bmatrix} \right\}. \end{aligned} \tag{19}$$

It follows now that the eigenvalues of A are the roots of the following characteristic equation

$$s^2 + b_0 s + k_0 \lambda_\ell = 0. \tag{20}$$

For each $\ell \in \{1, 2, \cdots, N\}$, the two roots of the characteristic equation are denoted by $s_\ell^\pm$,

$$s_\ell^\pm = \frac{-b_0 \pm \sqrt{b_0^2 - 4k_0\lambda_\ell}}{2}. \tag{21}$$

The *least stable* eigenvalue is the one closet to the imaginary axis among them, it is denoted by $s_{\min}$. It follows from Definition 2 that $S_o = |Re(s_{\min})|$.

Depending on the discriminant in (21), there are two cases to analyze:

1) If $\lambda_1 \geq 4k_0/b_0^2$, then the discriminant in (21) for each ℓ is non-positive, which yields

$$S_o = |Re(s_{\min})| = \frac{b_0}{2}.$$

2) Otherwise, the less stable eigenvalue can be written as $s_\ell^+ = \frac{-b_0 + \sqrt{b_0^2 - 4k_0\lambda_\ell}}{2}$. The least stable eigenvalue is obtained by setting $\lambda_\ell = \lambda_1$, so that

$$S_o = |Re(s_{\min})| = \frac{b_0 - \sqrt{b_0^2 - 4k_0\lambda_1}}{2}. \quad \blacksquare$$

We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. From Lemma 1, we see that the smallest eigenvalue of the directed grounded Laplacian plays an important role in determining the stability margin of the 1-D flock. To get an lower bound of the stability margin, a lower bound for the smallest eigenvalue is needed. For the general asymmetric case ($0 < \epsilon < 1$), it follows from Theorem 3.1 of [14] that the eigenvalues of L_g are given by

$$\lambda_\ell = 2 - 2\sqrt{1 - \epsilon^2} \cos\theta_\ell, \quad \ell \in \{1, 2, \cdots, N\}, \tag{22}$$

where $\epsilon \in (0,1)$ and θ_ℓ is the ℓ-th root of the following equation

$$\sqrt{\frac{1+\epsilon}{1-\epsilon}} \sin(N+1)\theta = \sin N\theta. \tag{23}$$

From formula (22), we see that the eigenvalues of L_g are real and positive, and moreover, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$. To see why, first notice that we only need consider the roots of (23) in the open interval $(0, 2\pi)$, in which there are $2N$ nontrivial isolated roots. The roots located in $\mathbb{R} \setminus (0, 2\pi)$ are $2m\pi$ ($m \in \mathbb{Z}$) distance away from those in $(0, 2\pi)$. Moreover, if $\theta_0 \in (0, 2\pi)$ is a solution of (23), then $2\pi - \theta_0$ is also a solution. Therefore, we can restrict the domain of analysis to $(0, \pi)$, in which there are N isolated roots. The ordering of the eigenvalues follows from $\cos\theta$ being a decreasing function in $(0, \pi)$.

It also follows that θ_1, the smallest positive root of (23), leads to the smallest eigenvalue. It is straightforward to see from the graphical solution of (23) that the ℓ-th root θ_ℓ is in the open interval $(\frac{(2\ell-1)\pi}{2(N+1)}, \frac{(2\ell+1)\pi}{2(N+1)})$. Now, the smallest eigenvalue of the directed grounded Laplacian L_g is given by

$$\lambda_1 = 2 - 2\sqrt{1-\epsilon^2}\cos\theta_1, \tag{24}$$

where $\theta_1 \in (\frac{\pi}{2(N+1)}, \frac{3\pi}{2(N+1)})$. Take the limit $N \to \infty$, we obtain the following infimum for the smallest eigenvalue λ_1

$$\inf_N \lambda_1 = 2 - 2\sqrt{1-\epsilon^2}. \tag{25}$$

To prove Theorem 1, we consider the following two cases:
1) $\lambda_1 \geq 4k_0/b_0^2$. According to Lemma 1, the stability margin is given by $S_o = b_0/2$.
2) $\lambda_1 < 4k_0/b_0^2$. From Lemma 1, the stability margin is given by $S_o = \frac{b_0 - \sqrt{b_0^2 - 4k_0\lambda_1}}{2}$. Since $\lambda_1 \geq 2 - 2\sqrt{1-\epsilon^2}$, the stability margin for this case is bounded below

$$S_o \geq \frac{b_0 - \sqrt{b_0^2 - 8k_0(1-\sqrt{1-\epsilon^2})}}{2}. \tag{26}$$

Notice that the above lower bound (26) is smaller than $b_0/2$ (value of S_o in case 1)). The real part sign $\Re(.)$ of (16) comes from by combining the above two cases. ■

A. Numerical comparisons

In this section, we present the numerical verification of the lower bound predicted by Theorem 1. The stability margins are obtained by numerically evaluating the eigenvalues of the state matrix A. In addition, we also compare the stability margins between symmetric and asymmetric controls. The nominal control gains used are $k_0 = 1$, $b_0 = 0.5$, and for asymmetric control, the amount of asymmetry used is $\epsilon = 0.1$.[2] We note that for asymmetric control, the control gains satisfy the second case of Lemma 1, so that the Theorem 1 predicts that the stability margin is bounded below by $(b_0 - \sqrt{b_0^2 - 8k_0(1-\sqrt{1-\epsilon^2})})/2 \approx 0.0209$. We can see from Figure 2 that the stability margin of the flock with asymmetric control is indeed bounded away from 0 uniformly in N, and the prediction (16) of Theorem 1 is quite accurate. Furthermore the stability margin with asymmetric control is much larger than that with symmetric control for the same N.

IV. Stability margin of the PDE approximation of flock dynamics

In this section, we present the stability margin of the flock with PDE model (13) and boundary condition (8). Since the PDE model (13) and boundary condition (8) are linear and homogeneous, we are able to apply the method of separation of variables. We assume a solution of the form $\tilde{p}(x,t) = \sum_{\ell=1}^{\infty} \phi_\ell(x) h_\ell(t)$. Substituting it into PDE (13), we obtain the following time-domain ODE

$$\frac{d^2 h_\ell(t)}{dt^2} + b_0 \frac{dh_\ell(t)}{dt} + k_0 \mu_\ell h_\ell(t) = 0, \tag{27}$$

[2]When ϵ is large, numerical errors in eigenvalue computations arise when the dimension of the matrix A is large. This is observed by numerically comparing the eigenvalues of the matrix with those of a random similarity transformation of the matrix, which in MATLAB© produces distinct results.

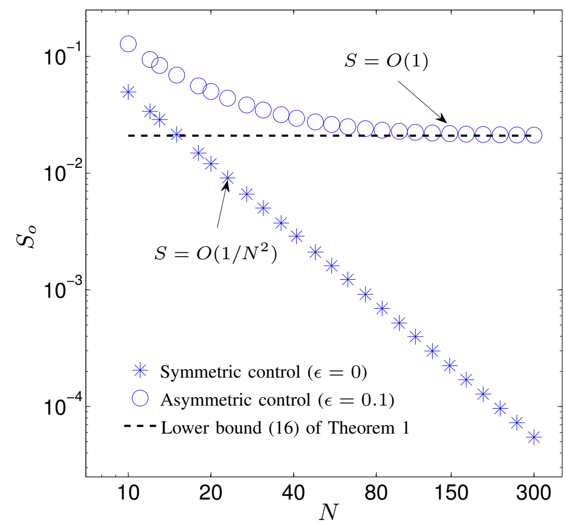

Fig. 2. Stability margin comparisons between the flock with symmetric control and asymmetric control.

where μ_ℓ solves the following boundary value problem

$$\mathcal{L}\phi_\ell(x) = 0, \quad \mathcal{L} := \frac{d^2}{dx^2} + 2\epsilon N \frac{d}{dx} + \mu_\ell N^2, \tag{28}$$

with the following boundary condition, which comes from (8):

$$\frac{d\phi_\ell}{dx}(0) = 0, \quad \phi_\ell(1) = 0. \tag{29}$$

Taking Laplace transform of both sides of (27) with respect to the time variable t, we have the following characteristic equation for the PDE model

$$s^2 + b_0 s + k_0 \mu_\ell = 0. \tag{30}$$

To prove Theorem 2, we need the following lemma.

Lemma 2: The eigenvalues μ_ℓ ($\ell \in \{1, 2, \cdots\}$) of the Strum-Liouville operator $\mathcal{L}$ (28) with boundary condition (29) for $0 < \epsilon < 1$ are real and satisfy

$$\mu_\ell = \epsilon^2 + \frac{a_\ell^2}{N^2}, \tag{31}$$

where a_ℓ is the root of $-a_\ell/(\epsilon N) = \tan(a_\ell)$, and in particular, $a_\ell \in (\frac{(2\ell-1)\pi}{2}, \ell\pi)$. □

Proof of Lemma 2. We first multiply both sides of (28) by $e^{2\epsilon N x} N^2$, we obtain the standard Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(e^{2\epsilon N x}\frac{d\phi_\ell(x)}{dx}\right) + \mu_\ell N^2 e^{2\epsilon N x}\phi_\ell(x) = 0. \tag{32}$$

According to Sturm-Liouville Theory, all the eigenvalues are real and have the following ordering $\mu_1 < \mu_2 < \cdots$, see [15]. To solve the boundary value problem (28)-(29), we assume solution of the form, $\phi_\ell(x) = e^{rx}$, then we obtain the following equation

$$r^2 + 2\epsilon N r + \mu_\ell N^2 = 0 \quad \Rightarrow \quad r = -\epsilon N \pm N\sqrt{\epsilon^2 - \mu_\ell}.$$

Depending on the discriminant in the above equation, there are two cases to analyze:

1) $\mu_\ell > \epsilon^2$, then the eigenfunction $\phi_\ell(x)$ has the following form $\phi_\ell(x) = e^{-\epsilon N x}(c_1 \cos(N\sqrt{\mu_\ell - \epsilon^2}x) + c_2 \sin(N\sqrt{\mu_\ell - \epsilon^2}x))$. Applying the boundary condition (29), for non-trivial eigenfunctions $\phi_\ell(x)$ to exit, the eigenvalues μ_ℓ must satisfy (31) and a_ℓ solves the transcendental equation $-a_\ell/(\epsilon N) = \tan(a_\ell)$. A graphical representation of the functions $\tan x$ and $-x/\epsilon N$ with respect to x shows that $a_\ell \in (\frac{(2\ell-1)\pi}{2}, \ell\pi)$.

2) $\mu_\ell \leq \epsilon^2$. Following the step of case 1), it's straightforward to show that there is no eigenvalue for this case. ■

We now present the proof for Theorem 2.

Proof of Theorem 2. From Lemma 2, we see that $a_1 \in (\pi/2, \pi)$, and (31) implies $\mu_1 \to \epsilon^2$ from above as $N \to \infty$, i.e. $\inf_N \mu_1 = \epsilon^2$. From the characteristic equation (30), the eigenvalues of the PDE model are given by

$$s_\ell^\pm = \frac{-b_0 \pm \sqrt{b_0^2 - 4k_0\mu_\ell}}{2}. \tag{33}$$

Depending on the discriminant in (33), there are two cases to analyze:

(1) If $\mu_1 \geq 4k_0/b_0^2$, then the discriminant in (33) for each ℓ is non-positive, which yields $S_p = |Re(s_{\min})| = b_0/2$.

(2) Otherwise, the less stable eigenvalue can be written as $s_\ell^+ = \frac{-b_0+\sqrt{b_0^2-4k_0\mu_\ell}}{2}$. The least stable eigenvalue is obtained by setting $\mu_\ell = \mu_1$, so that

$$S_p = |Re(s_{\min})| = \frac{b_0 - \sqrt{b_0^2 - 4k_0\mu_1}}{2} \geq \frac{b_0 - \sqrt{b_0^2 - 4k_0\epsilon^2}}{2}.$$

Again, note that the above lower bound is smaller than $b_0/2$ (value of S_p in case 1)). The real part sign $\Re(.)$ comes from by combining the above two cases. ■

V. Summary

We studied the stability margin of a large 1-D flock of double-integrator agents with decentralized control. Inspired by the previous works [1], [7], we examined the role of asymmetry in the control gains on the stability margin of the flock. We showed that with a fixed but non-vanishing amount of asymmetry in the control gains, the stability margin of the 1-D flock can be bounded away from 0 uniformly in N. This eliminates the problem of loss of stability margin with increasing N that is seen with symmetric control. Although we limit ourselves to 1-D flocks due to lack of space, extension to D-dimensional formations is straightforward and presented in [16].

Even though the asymmetric control design studied here (and in [16]) is the same as that in [1], [7], the conclusion is stronger: instead of $O(1/N)$ we get a $O(1)$ bound. The stronger result comes from avoiding the perturbation method used in [1], [7], which limited the analyses in those papers to vanishingly small amount of asymmetry. The results we obtained with the PDE approximation are slightly less accurate than those with analysis of coupled ODEs. This is to be expected since the PDE is an approximation of the coupled ODEs. An analysis of the approximation error will be presented elsewhere.

In this paper we consider a control law that requires relative position and absolute velocity feedback. However, even if we are constrained to use relative position and relative velocity feedback only, a similar size-independent stability margin can be achieved with asymmetric control, which is shown in [16].

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