Control of large 1D networks of double integrator agents: role of heterogeneity and asymmetry on stability margin

He Hao and Prabir Barooah

***Abstract*— We consider the distributed control of a network of heterogeneous agents with double integrator dynamics to maintain a rigid formation in 1D. The control signal at a vehicle is allowed to use relative position and velocity with its immediate neighbors. We examine the effect of heterogeneity and asymmetry on the closed loop stability margin, which is measured by the real part of the least stable eigenvalue. By using a PDE approximation, we show that heterogeneity has little effect while asymmetry has a significant effect on the stability margin. When control is symmetric, in which information from front and back neighbors are weighted equally, the stability margin decays to 0 as $O(1/N^2)$, where N is the number of agents, even when the agents are heterogeneous in their masses and control gains. In contrast, we show that arbitrarily small amount of asymmetry in the velocity feedback gains can improve the decay of the stability margin to $O(1/N)$. With equal amount of asymmetry in both velocity and position feedback gains, the closed loop is stable for arbitrary N. Numerical computations of the eigenvalues are provided that corroborate the PDE-based analysis.**

I. INTRODUCTION

In this paper we examine the closed loop dynamics of a system consisting of N interacting agents arranged in a line, where the agents are modeled as double integrators and each agent interacts with its nearest neighbors in the 1D network through its local control action. This is a problem that is of primary interest to formation control applications, especially to platoons of vehicles, where the vehicles are modeled as point masses. An extensive literature exists on 1D automated platoons; see [1], [2], [3] and references therein. In the vehicular platoon problem, each vehicle tries to maintain a constant gap between itself and its nearest neighbors. The desired trajectory of the entire network is available only to agent 1.

Although significant amount of research has been conducted on robustness-to-disturbance and stability issues of double integrator networks with decentralized control, most investigations consider the homogeneous case in which each agent has the same mass and employs the same controller (exceptions include [4], [5]). In addition, only symmetric control laws are considered in which the information from both the neighboring agents are weighted equally, with [6], [3] being exceptions. Khatir *et. al.* proposes heterogeneous control gains to improve string stability (sensitivity to disturbance) at the expense of control gains increasing without bound as N increases [4]. Middleton *et. al.* considers both unidirectional and bidirectional control, and concludes heterogeneity has little effect on the string stability under reasonable confines of bounded high frequency response and integral absolute error [5]. On the other hand, [6] examines the effect of asymmetry (but not heterogeneity) on the response of the platoon as a result of sinusoidal disturbances in the lead vehicle, and concludes the asymmetry makes sensitivity to such disturbances worse.

In this paper we analyze the case when the agents are *heterogeneous* in their masses and control laws used, and also allow asymmetry in the use of front and back information. A decentralized *bidirectional* control law is considered that uses only relative position and relative velocity information from the nearest neighbors. We examine the effect of heterogeneity and asymmetry on the stability margin of the closed loop, which is measured by the absolute value of the real part of the least stable eigenvalue. The stability margin determines the decay rate of initial formation keeping errors. Such errors arise from poor initial arrangement of the agents. The main result of the paper is that in a decentralized bidirectional control, heterogeneity has little effect on the stability margin of the overall closed loop, while even small asymmetry can have a significant impact. In particular, we show that in the symmetric case, the stability margin decays to 0 as $O(1/N^2)$, where N is the number of agents. We also show that the asymptotic trend of stability margin is not changed by agent-to-agent heterogeneity as long as the control gains do not have front-back asymmetry. On the other hand, arbitrary small amount of asymmetry in the way the local controllers use front and back information can improve the stability margin to $O(1/N)$! To achieve such an improvement, each agent has to weigh relative velocity information from its front neighbor more heavily than the one behind it.

Most of the results in this paper are established by using a PDE approximation of the coupled system of ODEs that model the closed loop dynamics of the network. Compared to previous work [3], this paper makes two novel contributions. First, we consider heterogeneous agents (the mass and control gains vary from agent to agent), whereas [3] consider only homogeneous agents. Secondly, [3] considered the scenario in which the desired trajectory of the platoon was one with a constant velocity, and moreover, every agent knew this desired velocity. In contrast, the control law we consider requires agents to know only the desired inter-agent separation; the overall trajectory information is made available only to agent 1. This makes the model more applicable to practical formation control applications

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in which the formation may be required to accelerate or decelerate occasionally, and the decision to do so is made solely by the lead agent. It was shown in [3] for the homogeneous formation that asymmetry in the position feedback can improve the stability margin from $O(1/N^2)$ to $O(1/N)$ while the absolute velocity feedback gain did not affect the asymptotic trend. In contrast, we show in this paper that with relative position and velocity feedback, asymmetry in the velocity feedback gain is the most significant determinant. Even small amount of asymmetry, when properly chosen, can lead to significant improvement in stability margin. With equal amount of asymmetry in both position and velocity feedback, the closed loop is stable for arbitrary N. The effect of asymmetry in position feedback alone is an open question, as is the general case of unequal asymmetry in position and velocity gains.

Although the PDE approximation is valid only in the limit $N \to \infty$, numerical comparisons with the original state-space model shows that the PDE model provides accurate results even for small N (5 to 10). PDE approximation is quite common in many-particle systems analysis in statistical physics and traffic-dynamics; see the article [8] for an extensive review. The usefulness of PDE approximation in analyzing multi-agent coordination problems has been recognized also by researchers in the controls community; see [9], [10], [3], [11] for examples. A similar but distinct framework based on partial *difference* equations has been developed by Ferrari-Trecate *et. al.* in [12]

The rest of this paper is organized as follows. Section II presents the problem statement. Section III describes the PDE model of the network of agents. Analysis and control design results together with their numerical corroboration appear in Sections IV and V, respectively. The paper ends with a summary in Section VI.

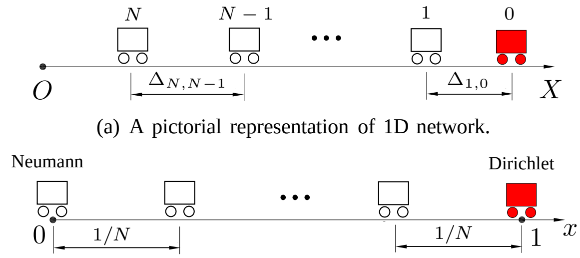
II. Problem Statement

We consider the formation control of N heterogeneous agents which are moving in 1D Euclidean space, as shown in Figure 1 (a). The position and mass of each agent are denoted by $p_i \in \mathbb{R}$ and m_i respectively. The mass of each agent is bounded, $|m_i - m_0|/m_0 \leq \delta$ for all i, where $m_0 > 0$ and $\delta \in [0, 1)$ are constants. The dynamics of each agent are modeled as a double integrator:

$$m_i \ddot{p}_i = u_i, \tag{1}$$

where u_i is the control input.

The information on the desired trajectory of the network is provided to agent 1. We introduce a *fictitious* reference agent with index 0 that perfectly tracks its desired trajectory, which is denoted by $p_0^*(t)$. Agent 1 is allowed to communicate with the reference agent. The desired geometry of the formation is specified by the *desired gaps* $\Delta_{i,i-1}$ for $i = 1, \dots, N$, where $\Delta_{i,i-1}$ is the desired value of $p_{i-1}(t) - p_i(t)$. The control objective is to maintain a rigid formation, i.e., to make neighboring agents maintain their pre-specified desired gaps and to make agent 1 follow its desired trajectory $p_0^*(t) - \Delta_{1,0}$.



(a) A pictorial representation of 1D network.

(b) A Redrawn graph of the same network.

Fig. 1. Desired geometry of a network with N agents and 1 "reference agent", which are moving in 1D Euclidean space. The filled agent in the front of the network represents the reference agent, it is denoted by "0". (a) is the original graph of the network in the $p \in [0, \infty)$ coordinate and (b) is the redrawn graph of the same network in the $\tilde{p} \in [0, 1]$ coordinate.

Since we are only interested in maintaining rigid formations that do not change shape over time, $\Delta_{i,i-1}$'s are positive constants.

In this paper, we consider the following *decentralized, bidirectional* control law, whereby the control action at the i-th agent depends on relative position and velocity measurements with its immediate neighbors in the 1D network:

$$\begin{aligned} u_i = &- k_i^f(p_i - p_{i-1} + \Delta_{i,i-1}) - k_i^b(p_i - p_{i+1} - \Delta_{i+1,i}) \\ &- b_i^f(\dot{p}_i - \dot{p}_{i-1}) - b_i^b(\dot{p}_i - \dot{p}_{i+1}), \end{aligned} \tag{2}$$

where $i = \{1, \dots, N-1\}$. $k_{(.)}^f, k_{(.)}^b$ are the front and back position gains and $b_{(.)}^f, b_{(.)}^b$ are the front and back velocity gains respectively. For the agent with index N which does not have an agent behind it, the control law is slightly different:

$$u_i = - k_i^f(p_i - p_{i-1} + \Delta_{i,i-1}) - b_i^f(\dot{p}_i - \dot{p}_{i-1}). \tag{3}$$

Each agent i knows the desired gaps $\Delta_{i,i-1}$ and $\Delta_{i+1,i}$, while only agent 1 knows the desired trajectory $p_0^*(t)$ of the fictitious reference agent. Combining the open loop dynamics (1) with the control law (2), we get

$$\begin{aligned} m_i \ddot{p}_i = &- k_i^f(p_i - p_{i-1} - \Delta_{i,i-1}) \\ &- k_i^b(p_i - p_{i+1} - \Delta_{i,i+1}) \\ &- b_i^f(\dot{p}_i - \dot{p}_{i-1}) - b_i^b(\dot{p}_i - \dot{p}_{i+1}), \end{aligned} \tag{4}$$

where $i \in \{1, \dots, N-1\}$. The dynamics of the N-th agent are obtained by combining (1) and (3), which are slightly different from (4). The desired trajectory of the i-th agent is $p_0^*(t) - \sum_{j=i}^{1} \Delta_{j,j-1} =: p_i^*(t)$. To facilitate analysis, we define the tracking error:

$$\tilde{p}_i := p_i - p_i^* \qquad \Rightarrow \qquad \dot{\tilde{p}}_i = \dot{p}_i - \dot{p}_i^*. \tag{5}$$

Substituting (5) into (4), and using $p_{i-1}^*(t) - p_i^*(t) = \Delta_{i,i-1}$, we get

$$\begin{aligned} m_i \ddot{\tilde{p}}_i = &- k_i^f(\tilde{p}_i - \tilde{p}_{i-1}) - k_i^b(\tilde{p}_i - \tilde{p}_{i+1}) \\ &- b_i^f(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i-1}) - b_i^b(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i+1}), \end{aligned} \tag{6}$$

where we have used the fact that $\tilde{p}_0(t) \equiv 0$ since the trajectory of the reference agent is equal to its desired trajectory.

By defining the state $\psi := [\tilde{p}_1, \dot{\tilde{p}}_1, \tilde{p}_2, \dot{\tilde{p}}_2, \cdots, \tilde{p}_N, \dot{\tilde{p}}_N]^T$, the closed loop dynamics of the network can now be written compactly from (6) as:

$$\dot{\psi} = \mathbf{A}\psi \tag{7}$$

where $\mathbf{A}$ is the closed-loop state matrix. The stability margin of the network, which is denoted by S, is defined as the absolute value of the real part of the least stable eigenvalue of $\mathbf{A}$. In this paper, most of analysis and design is performed using a PDE approximation of the state space model (7), which is described next.

III. PDE MODEL OF THE NETWORK

We now derive a continuum approximation of the closed loop dynamics (7) in the limit of large N, by following the steps involved in a finite-difference discretization in reverse. We define $k_i^{f+b} := k_i^f + k_i^b$, $k_i^{f-b} := k_i^f - k_i^b$, $b_i^{f+b} := b_i^f + b_i^f$, $b_i^{f-b} := b_i^f - b_i^b$. Substituting these into (6),

$$\begin{aligned} &m_i \ddot{\tilde{p}}_i = \\ &-\frac{k_i^{f+b} + k_i^{f-b}}{2}(\tilde{p}_i - \tilde{p}_{i-1}) - \frac{k_i^{f+b} - k_i^{f-b}}{2}(\tilde{p}_i - \tilde{p}_{i+1}) \\ &-\frac{b_i^{f+b} + b_i^{f-b}}{2}(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i-1}) - \frac{b_i^{f+b} - b_i^{f-b}}{2}(\dot{\tilde{p}}_i - \dot{\tilde{p}}_{i+1}). \end{aligned} \tag{8}$$

To facilitate analysis, we redraw the graph of the 1D network, so that the position error $\tilde{p}_i$ are defined in the interval $[0,1]$, irrespective of the number of agents. The i-th agent in the "original" graph, is now drawn at position $(N-i)/N$ in the new graph. Figure 1 shows an example.

The starting point for the PDE derivation is to consider a function $\tilde{p}(x,t) : [0,1] \times [0,\ \infty) \to \mathbb{R}$ that satisfies:

$$\tilde{p}_i(t) = \tilde{p}(x,t)|_{x=(N-i)/N}, \tag{9}$$

such that functions that are defined at discrete points i will be approximated by functions that are defined everywhere on $[0,1]$. The original functions are thought of as samples of their continuous approximations. We formally introduce the following scalar functions $k^f(x), k^b(x), b^f(x), b^b(x)$ and $m(x) : [0,1] \to \mathbb{R}$ defined according to the stipulation:

$$\begin{aligned} k_i^f &= k^f(x)|_{x=(N-i)/N}, \quad & k_i^b &= k^b(x)|_{x=(N-i)/N}, \\ b_i^f &= b^f(x)|_{x=(N-i)/N}, \quad & b_i^b &= b^b(x)|_{x=(N-i)/N}, \\ m_i &= m(x)|_{x=(N-i)/N}. \end{aligned} \tag{10}$$

In addition, we define functions $k^{f+b}(x)$, $k^{f-b}(x)$, $b^{f+b}(x)$, $b^{f-b}(x) : [0,1]^D \to \mathbb{R}$ as

$$\begin{aligned} k^{f+b}(x) &:= k^f(x) + k^b(x), \quad & k^{f-b}(x) &:= k^f(x) - k^b(x), \\ b^{f+b}(x) &:= b^f(x) + b^b(x), \quad & b^{f-b}(x) &:= b^f(x) - b^b(x). \end{aligned}$$

Due to (10), these satisfy

$$\begin{aligned} k_i^{f+b} &= k^{f+b}(x)|_{x=(N-i)/N}, \quad & k_i^{f-b} &= k^{f-b}(x)|_{x=(N-i)/N} \\ b_i^{f+b} &= b^{f+b}(x)|_{x=(N-i)/N}, \quad & b_i^{f-b} &= b^{f-b}(x)|_{x=(N-i)/N}. \end{aligned}$$

To obtain a PDE model from (8), we first rewrite it as

$$\begin{aligned} m_i \ddot{\tilde{p}}_i = &\frac{k_i^{f-b}}{N}\frac{(\tilde{p}_{i-1} - \tilde{p}_{i+1})}{2(1/N)} + \frac{k_i^{f+b}}{2N^2}\frac{(\tilde{p}_{i-1} - 2\tilde{p}_i + \tilde{p}_{i+1})}{1/N^2} \\ &\frac{b_i^{f-b}}{N}\frac{(\dot{\tilde{p}}_{i-1} - \dot{\tilde{p}}_{i+1})}{2(1/N)} + \frac{b_i^{f+b}}{2N^2}\frac{(\dot{\tilde{p}}_{i-1} - 2\dot{\tilde{p}}_i + \dot{\tilde{p}}_{i+1})}{1/N^2}. \end{aligned} \tag{11}$$

Using the following finite difference approximations for every $i \in \{1, \ldots, N-1\}$:

$$\begin{aligned} \Big[\frac{\tilde{p}_{i-1} - \tilde{p}_{i+1}}{2(1/N)}\Big] &= \Big[\frac{\partial \tilde{p}(x,t)}{\partial x}\Big]_{x=(N-i)/N}, \\ \Big[\frac{\tilde{p}_{i-1} - 2\tilde{p}_i + \tilde{p}_{i+1}}{1/N^2}\Big] &= \Big[\frac{\partial^2 \tilde{p}(x,t)}{\partial x^2}\Big]_{x=(N-i)/N}, \\ \Big[\frac{\dot{\tilde{p}}_{i-1} - \dot{\tilde{p}}_{i+1}}{2(1/N)}\Big] &= \Big[\frac{\partial^2 \tilde{p}(x,t)}{\partial x \partial t}\Big]_{x=(N-i)/N}, \\ \Big[\frac{\dot{\tilde{p}}_{i-1} - 2\dot{\tilde{p}}_i + \dot{\tilde{p}}_{i+1}}{1/N^2}\Big] &= \Big[\frac{\partial^3 \tilde{p}(x,t)}{\partial x^2 \partial t}\Big]_{x=(N-i)/N}, \end{aligned}$$

Eq. (11) is seen as a finite difference approximation of the following PDE:

$$\begin{aligned} m(x)\Big(\frac{\partial^2}{\partial t^2}\Big)\tilde{p}(x,t) = \Big(&\frac{k^{f-b}(x)}{N}\frac{\partial}{\partial x} + \frac{k^{f+b}(x)}{2N^2}\frac{\partial^2}{\partial x^2} + \\ &\frac{b^{f-b}(x)}{N}\frac{\partial^2}{\partial x \partial t} + \frac{b^{f+b}(x)}{2N^2}\frac{\partial^3}{\partial x^2 \partial t}\Big)\tilde{p}(x,t). \end{aligned} \tag{12}$$

The boundary conditions of PDE (12) depend on the arrangement of reference agent in the information graph. For our case, the boundary conditions are of the Dirichlet type at $x=1$ where the reference agent is, and Neumann at $x=0$:

$$\tilde{p}(1,t) = 0, \qquad \frac{\partial \tilde{p}}{\partial x}(0,t) = 0. \tag{13}$$

IV. ROLE OF HETEROGENEITY ON STABILITY MARGIN

We first study the role of heterogeneity on the stability margin of the network with symmetric control.

Definition 1: The control law (2) is *symmetric* if each agent uses the same front and back control gains: $k_i^f = k_i^b, b_i^f = b_i^b$, for all $i \in \{1, 2, \ldots, N-1\}$. □

The first main result of the paper is the following.

Theorem 1: Consider an N-agent heterogeneous network with dynamics (1) and control law (2), (3), where the mass and the control gains of each agent satisfy $|m_i - m_0|/m_0 \leq \delta$, $|k_i^{(\cdot)} - k_0|/k_0 \leq \delta$ and $|b_i^{(\cdot)} - b_0|/b_0 \leq \delta$ where m_0, k_0 and b_0 are positive constants, and $\delta \in [0,1)$ denotes the amount of heterogeneity. With symmetric control, the stability margin S of the network satisfies the following:

$$(1-2\delta)\frac{\pi^2 b_0}{8m_0}\frac{1}{N^2} \leq S \leq (1+2\delta)\frac{\pi^2 b_0}{8m_0}\frac{1}{N^2}, \tag{14}$$

when $\delta \ll 1$. □

The result above is also provable for an arbitrary $\delta < 1$ (not necessarily small) when there is only heterogeneity in mass using standard results on Sturm-Liouville theory [13]. For that case, the result is given in the following lemma, and its proof is provided in [7].

Lemma 1: Consider an N-agent heterogeneous network with dynamics (1) and control law (2), (3), where the mass and the control gains of each agent satisfy $0 < m_{\min} \leq m_i \leq m_{\max}$, $k_i^f = k_i^b = k_0$ and $b_i^f = b_i^b = b_0$, where m_0, k_0 and b_0 are positive constants. The stability margin S of the network satisfies the following:

$$\frac{\pi^2 b_0}{8m_{\max}}\frac{1}{N^2} \leq S \leq \frac{\pi^2 b_0}{8m_{\min}}\frac{1}{N^2}. \tag{15}$$

□

The main implication of the above results is that *heterogeneity of masses and control gains plays no role in the asymptotic trend of the stability margin with N as long as the control gains are symmetric*. Note that the $O(1/N^2)$ decay of the stability margin described above has been shown for homogeneous platoons (all agents have the same mass and use the same control gains) independently in [14].

To prove Theorem 1, the starting point of our analysis is the investigation of the homogeneous and symmetric case: $m_i = m_0, k_i^{(\cdot)} = k_0, b_i^{(\cdot)} = b_0$ for some positive constants m_0, k_0, b_0, for $i \in \{1, \dots, N\}$. Using the notation introduced earlier, we get $m(x) = m_0, k^{f+b}(x) = 2k_0, k^{f-b}(x) = 0, b^{f+b}(x) = 2b_0, b^{f-b}(x) = 0$.

The PDE (12) simplifies to:

$$m_0\frac{\partial^2\tilde{p}(x,t)}{\partial t^2} = \frac{k_0}{N^2}\frac{\partial^2\tilde{p}(x,t)}{\partial x^2} + \frac{b_0}{N^2}\frac{\partial^3\tilde{p}(x,t)}{\partial x^2\partial t}. \tag{16}$$

This is a wave equation with Kelvin-Voight damping. Taking a Laplace transform of the above equation, we get

$$(m_0 s^2 - \frac{b_0 s + k_0}{N^2}\frac{\partial^2}{\partial x^2})\eta(s,x) = 0 \tag{17}$$

where $\eta(x,s)$ is the Laplace transform of $\tilde{p}(x,t)$. $\phi_\ell(x) = cos(\frac{2\ell-1}{2}\pi x)$ is the ℓ-th eigenfunction of the Laplacian $\frac{\partial^2}{\partial x^2}$ with the boundary condition $\eta(1,s) = 0, \frac{\partial}{\partial x}\eta(0,s) = 0$, which come from the boundary condition (13). The associated eigenvalues are

$$\lambda_\ell = \pi^2\frac{(2\ell-1)^2}{4}, \quad \ell = 1, 2, \dots \tag{18}$$

Plugging the expansion $\eta(x,s) = \sum_{\ell=1}^{\infty}\phi_\ell(x)\beta_\ell(s)$, where β_ℓ are weights into (17), we get the characteristic equation $m_0 s^2 + \frac{b_0 s + k_0}{N^2}\lambda_\ell = 0$, so that the eigenvalues of the PDE are

$$s_\ell^{\pm} = -\frac{\lambda_\ell b_0}{2m_0 N^2} \pm \frac{1}{2m_0 N}\sqrt{\frac{\lambda_\ell^2 b_0^2}{N^2} - 4\lambda_\ell m_0 k_0}. \tag{19}$$

For small ℓ and large N so that $N > (2\ell-1)\pi b_0/(4\sqrt{m_0 k_0})$, the discriminant is negative, making the real part of the eigenvalues equal to $-\lambda_\ell b_0/(2m_0 N^2)$. The least stable eigenvalue, the one closest to the imaginary axis, is obtained with $\ell = 1$:

$$s_1^{\pm} = -\frac{\pi^2 b_0}{8m_0}\frac{1}{N^2} \Rightarrow S = \frac{\pi^2 b_0}{8m_0 N^2}. \tag{20}$$

Now recall that for a heterogeneous network with symmetric control we have

$$k_i^f = k_i^b, \quad b_i^f = b_i^b, \quad \forall i \in \{1, \dots, N\}.$$

In this case, using the notation introduced earlier, we have

$$k^{f-b}(x) = 0, \quad b^{f-b}(x) = 0,$$

The PDE (12) is simplified to:

$$m(x)\frac{\partial^2\tilde{p}(x,t)}{\partial t^2} = \frac{k^{f+b}(x)}{2N^2}\frac{\partial^2\tilde{p}(x,t)}{\partial x^2} + \frac{b^{f+b}(x)}{2N^2}\frac{\partial^3\tilde{p}(x,t)}{\partial x^2\partial t}. \tag{21}$$

The proof of Theorem 1 follows from a perturbation analysis of the eigenvalues of the above PDE starting with the results of homogeneous and symmetric case. The interested reader is referred to [7] for the details of the proof.

A. Numerical comparison

We now present numerical computations that corroborates the PDE-based analysis. We consider the following mass and control gain profile:

$$\begin{aligned} k_i^f = k_i^b &= 1 + 0.2\sin(2\pi(N-i)/N), \\ b_i^f = b_i^b &= 0.5 + 0.1\sin(2\pi(N-i)/N), \\ m_i &= 1 + 0.2\sin(2\pi(N-i)/N). \end{aligned} \tag{22}$$

In the associated PDE model (21), this corresponds to $k^{f+b}(x) = 2 + 0.4\sin(2\pi x)$, $b^{f+b}(x) = 1 + 0.2\sin(2\pi x)$, $m(x) = 1 + 0.2\sin(2\pi x)$. The eigenvalues of the PDE, that are computed numerically using a Galerkin method with Fourier basis, are compared with that of the state space model to check how well the PDE model captures the closed loop dynamics. The interested reader is referred to [7] for detailed comparisons. Here we only present comparisons for the least stable eigenvalue, which is shown in Figure 2 as a function of N. We see from Figure 2 that the closed-loop stability margin of the controlled formation is well captured by the PDE model. In addition, the plot corroborates the predicted bound (14).

V. Role of asymmetry on stability margin

The second main result of this work is that the stability margin can be greatly improved by introducing front-back asymmetry in the velocity-feedback gains. We call the resulting design *mistuning*-based design because it relies on small changes from the nominal symmetric gain b_0. In addition, a poor choice of such asymmetry can also make the closed loop unstable.

Theorem 2: For an N-agent network with dynamics (1) and control law (2), (3), with $m_i = m_0$ for all i, consider the problem of maximizing the stability margin by choosing the control gains with the constraint $|b_i^{(\cdot)} - b_0|/b_0 \leq \varepsilon$ for all i, with ε being a positive constant, and $k_i^{(f)} = k_i^{(b)} = k_0$. For vanishingly small values of ε, the optimal gains are

$$b_i^f = (1+\varepsilon)b_0, \qquad b_i^b = (1-\varepsilon)b_0, \tag{23}$$

which result in the stability margin

$$S = \frac{\varepsilon b_0}{m_0}\frac{1}{N} + O(\frac{1}{N^2}). \tag{24}$$

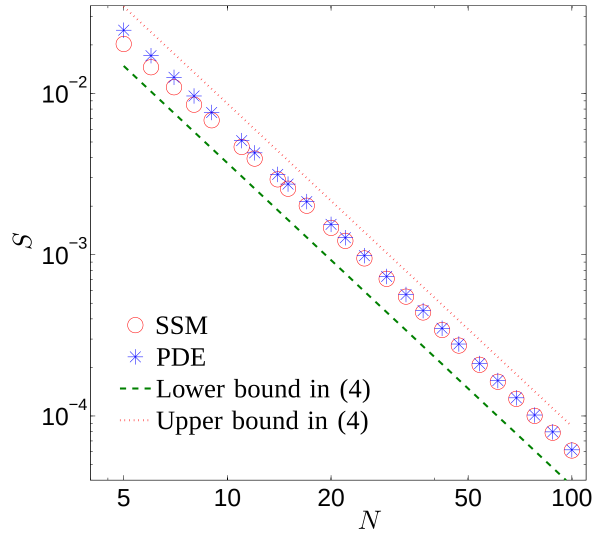


Fig. 2. The stability margin of the heterogeneous formation with symmetric control as a function of number of agents: the legends of SSM, PDE and lower bound, upper bound stand for the stability margin computed from the state space model, from the PDE model, and the asymptotic lower and upper bounds (14) in Theorem 1. The mass and control gains profile are given in (22).

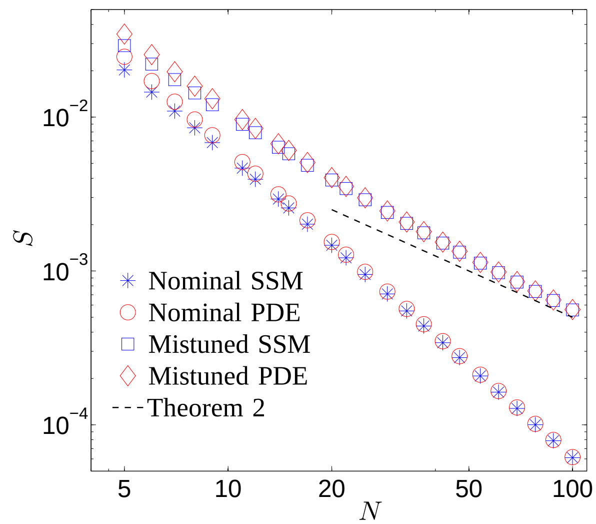


Fig. 3. Stability margin improvement by mistuning design. The nominal control gains are $k_0 = 1$, $b_0 = 0.5$, and the mistuned gains used are the ones given by (23) in Theorem 2 with $\varepsilon = 0.1$. The legends "Nominal SSM" and "Nominal PDE" stand for the stability margin computed from the state-space model and the PDE model, respectively, with symmetric control. The legends "Mistuned SSM" and "Mistuned PDE" stand for the stability margin computed from the state-space model and PDE model, respectively, with mistuned control.

The formula is asymptotic in the sense that it holds when $N \to \infty$ and $\varepsilon \to 0$. □

Similar to Theorem 1, the proof of Theorem 2 also relies on a perturbation method. Please refer to [7] for the details of the proof.

The theorem says that with arbitrary small change in the front-back asymmetry in velocity feedback, so that the velocity information from the front is weighted more heavily than the one from the back, the stability margin improves significantly.

The astute reader may inquire at this point what are the effects of introducing asymmetry in the position-feedback gains while keeping velocity gains symmetric, or introducing asymmetry in both position and velocity feedback gains. It turns out when equal asymmetry in both position and velocity feedback gains are introduced, the closed loop is stable for arbitrary N. We state the result in the next theorem.

Theorem 3: The closed loop dynamics of the N-agent network with the following asymmetry in control $k_i^f = (1+\rho)k_0$, $k_i^b = (1-\rho)k_0$, $b_i^f = (1+\rho)b_0$, $b_i^b = (1-\rho)b_0$, where ρ is a constant satisfying $\rho \in (-1, 1]$, are exponentially stable. □

The result above is for a equal amount (as a fraction of the nominal value) of asymmetry in the position feedback and velocity feedback gains. This constraint of equal asymmetry in position and velocity feedback is imposed in order to make the analysis tractable. Veerman proved a very similar result [6, Theorem 4.2], though the model was slightly different: the N-th agent's control law was $u_N = k^f(p_{N-1} - p_N) - b^f(\dot{p}_{N-1} - \dot{p}_N)$. Our proof (provided in [7]) follows a similar line of attack, by analyzing the role of asymmetry on the least stable eigenvalue of state matrix $\mathbf{A}$.

The analysis of the stability margin in the following cases are open problems: (i) unequal asymmetry in position and velocity feedback, (ii) velocity feedback gains are kept at their nominal symmetric values and asymmetry is introduced in the position feedback gains only.

A. Comparison of stability margin computed from mistuned state-space and PDE models

Figure 3 depicts the numerically obtained mistuned and nominal stability margins for both the PDE and state-space models. The nominal control gains are $k_0 = 1$, $b_0 = 0.5$, and the mistuned velocity gains used are the ones given by (23) in Theorem 2 with $\varepsilon = 0.1$.

The figure shows that i) the closed-loop least stable eigenvalue match the PDE's accurately, even for small values of N; ii) the mistuned eigenvalues show large improvement over the symmetric case even though the velocity gains differ from their nominal values only by $\pm 10\%$. The improvement is particularly noticeable for large values of N, while being significant even for small values of N.

For comparison, the figure also depicts the asymptotic eigenvalue formula given in Theorem 2. The improvement in the stability margin with mistuning is remarkable even the velocity gains are changed from their symmetric values by only $\pm 10\%$. Another interesting aspect of the result in Theorem 2 is that the improvement from $O(1/N^2)$ to $O(1/N)$ can be achieved by *arbitrarily small changes* to the nominal velocity gains.

VI. Summary

We studied the role of heterogeneity and control asymmetry on the stability margin of a large $1D$ network of double-integrator agents. The control is decentralized; the control signal at every agent depends on the relative position and velocity measurements from its nearest neighbors. It is shown that heterogeneity does not effect how the stability margin scales with N, the number of agents, whereas asymmetry plays a significant role. As long as control is symmetric, meaning information on relative position and velocity from both neighbors are weighed equally, agent-to-agent heterogeneity does not change the $O(1/N^2)$ scaling of stability margin. If front-back asymmetry is introduced in the velocity feedback gains, even by an arbitrarily small amount, the stability margin can be improved to $O(1/N)$. This is a significant improvement, especially for large N. In addition, the optimal asymmetric (mistuned) gain profile is quite simple to implement. With a maximum allowable variation of $\pm 10\%$ from the symmetric velocity gains, the optimal gains are obtained by letting the front gains to be 10 percent larger than the nominal gain and letting the back gains to be 10 percent smaller.

The general case of asymmetry in both position and velocity feedback gains is an open problem. Some preliminary answers are available in the special case when equal amount of asymmetry in both position and velocity feedback is introduced, which is parameterized by ρ. We showed that in this case the closed loop is exponentially stable for arbitrary N. The stability margin as a function of ρ is also an open problem.

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