Calculation of Steady-State Probabilities of M/M Queues: Further Approaches

A assignment report submitted

for the course

Queuing Theory and Applications (MA597)

by

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0.1 Introduction

The M/M family of queues is generally the first to be introduced to students of operational research and queueing theory. The assumptions made in the model are that customers arrive randomly at a service point at a known rate λ per unit time, wait until a server is available and then require a random time with an exponential distribution with mean $1/\mu$ for service. It is possible to calculate the steady state probabilities for the number of customers in the queue system for many variants of this family of queues.

Here the problem arose when circumstances were encountered where the obvious approaches to solving the queueing models proved liable to severe rounding error.

0.2 Rounding error

In most references to computational accuracy, computer programmers are warned about the risks of rounding errors, and particularly truncation of small numbers to zero. For instance, the first chapter of Knuth [1], a text familiar to many students of computer science, works through an example of how rounding errors quickly lead to inaccurate results in a computational algorithm. Numerous examples can be cited of the need to avoid particular types of calculation, known to cause difficulties. A simple instance is the calculation of the sum and the individual terms of a geometric series, $S_N = \sum_0^N x^n$ for $|x| < 1$. Leaving aside the possibility of finding the sum analytically, the calculation may be organized from left to right or from right to left. The first would calculate the first term, and then subsequent terms by multiplying by x; the second would calculate the last term, and then use division by x to find the others. The second method relies on the accurate calculation and storage of x_N, which is liable to rounding errors, depending on the values of x and N. For small values of x and large N, say $x = 0.1$ and $N = 100$, x_N may be too small to be treated as distinct from zero. This rounding may not be registered by the computer, so that the user will be ignorant of the inaccuracy of calculations.

0.3 Calculating the Steady State Probabilities of Queues

This problem can be encountered when attempting to find the steady state probabilities of busy multi-server queues.

Given an $M/M/C/K$ queue (that is Poisson process arrivals, exponential service time, C servers and space for $K \geq C$ customers in the system), the steady state probabilities can be readily derived to be:

$$p_0 = 1/Q$$
$$p_1 = \frac{\lambda}{\mu} p_0$$
$$p_2 = \frac{\lambda}{\mu} \frac{p_1}{2}$$

....

$$p_C = \frac{\lambda}{\mu} \frac{p_C - 1}{C}$$
$$p_{C+1} = \frac{\lambda}{C\mu} p_C$$

$$p_K = \frac{\lambda}{C\mu} p_{K-1}$$
$$Q = 1 + \sum_{r=1}^{C} \frac{\lambda^r}{r!\mu^r} + \sum_{r=C+1}^{K} \frac{\lambda^r}{C!\mu^r C^{r-C}}$$

where λ is the arrival rate and μ the service rate. In many circumstances, the order of calculations will be to find Q first of all, then to take its reciprocal to obtain p_0, and then make use of the recurrence relationship to find all the steady state probabilities.

For an $M/M/30/40$ queue, which might be appropriate for modelling a telephone call centre, $\lambda = 10\mu$ would give $p_0 = 0.00005$, rounded up from six-figure accuracy of 0.000045. Use of the formula leads to the absurd position that the sum of all the steady state probabilities is more than 1, since all the terms in the sum will be scaled by the constant multiplier 0.00005/0.000045 or an increase of 11%. Increasing λ to 15μ gives $p_0 = 0.0$ to five decimal places, and so all the steady state probabiliies would be treated as zero. Other values of the queue parameters could lead to the rounded value of p0 being almost twice, or about two-thirds, of its correct value. In order to try and avoid these rounding errors, one could adopt the method employed for calculating the sum of the geometric series earlier, and start with the last term, which is the probability that the system is full. The order of calculation would be essentially the reverse of that given earlier.

$$p_K = 1/Q'$$
$$p_{K-1} = \frac{C\mu}{\lambda} p_K$$

....

$$p_0 = \frac{\mu}{\lambda} p_1$$

$$Q' = 1 + \sum_{r=1}^{C} \frac{r!\mu^r}{\lambda^r} + \sum_{r=C+1}^{K} \frac{C!\mu^r C^{r-C}}{\lambda^r}$$

Truncation error may mean that the value of p_K is zero, or is rounded to be significantly different from its correct value. For instance, with $\lambda = 20\mu$, use of Q' leads to 1.1% error in the evaluation of the probabilities, and with $\lambda = 15\mu$, $1/Q'$ is zero when rounded to five decimal places.

0.4 Conclusion

The errors may arise when finding steady state probabilities for queues in the $M/M/C/K$ family, and has demonstrated a safer approach to such calculation. The approach is easy to code into a spreadsheet, and can be used with a wide range of elementary queues.

Bibliography

[1] Knuth DE (1968), *The Art of Computer Programming. Volume 1: Fundamental Algorithms. Addison-Wesley, London.*

[2] Pasternack, BA and Drezner, Z (1998), *A Note on Calculating Steady State Results for an M/M/k Queueing System when the Ratio of the Arrival Rate to the Service Rate is Large. Journal of Applied Mathematics and Decision Sciences,* 2(2) p201203

[3] Taha HA (1992), *Operations Research: an introduction (5th edition).* Macmillan, New York

[4] Tijms HC (1986), *Stochastic Modelling and Analysis: a Computational Approach.* Wiley, Chichester, UK .