A Queueing Model for Meteor Burst Packet Communication Systems

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Abstract—**A discrete time queueing model for the performance of meteor burst packet communication systems is analyzed using matrix analytic methods. A meteor burst system uses ionized layers formed by naturally occurring meteor bursts in the earth's atmosphere to reflect radio signals. Due to its nuclear survivability, inherent privacy, and low cost, such a method of communication has gained considerable interest especially in the defence community. Not only is the system subject to interruptions due to the intermittence of the ionization layer, but its analysis is further complicated by the necessity to retransmit packets that have error or that occur at the tail end of a period of availability of the system. Our model takes such complexities into account. It is also of independent methodological interest in that it provides an exact analysis of a general queueing model with service interruptions. For the application at hand, we demonstrate the feasibility of the algorithms by a selected set of numerical examples. The algorithm can be used to ascertain the effects of the packet size, the bit error rate, the sync acquisition time, and other variables on system performance. A particularly useful aspect of our model is that it allows for the direct use of empirical distributions obtained from observational data.**

I. INTRODUCTION

A RATHER unusual approach to data transmission using ionization layers created in the earth's atmosphere by naturally occurring meteorite showers has gained importance due to many of its inherent advantages. Particularly in the military community, interest in such "meteor burst communication systems" is increasing [1] due to its nuclear survivability, inherent privacy, and low cost. Indeed, the Defence Communications Agency, the Department of Agriculture, the Department of Energy, and the National Oceanic and Atmospheric Administration have sponsored related studies. A number of such systems have been built, and among the applications are Beyond Line-of-Sight (BLOS) communications, sensor data collection, facsimile transmission, etc. For a detailed discussion of the basic facts concerning meteor burst systems, refer to [2] and [12].

In such a system, ionization layers created by meteorite showers are used to bounce radio signals thereby letting such layers fulfill the task of a communication satellite. For this reason, the system has been dubbed by some as the "poor man's satellite." Although meteorite showers arrive frequently, the useful life time of the ionization layer created by each shower is very short. It has been observed that while such showers occur at average intervals of about 10 s, the resulting layers provide effective transmission periods of only about 0.5 s; see [12] for a table of data pertaining to different seasons and times of the day. Thus, in queueing theory parlance, what we have is an intermittently available server.

In using this intermittently available medium for point to point packet communication from, say, A to B, a continuous probing signal (on a different frequency) is sent from B to A. Receipt of the probing signal by A signifies the beginnning of a period when the medium is available (operating period), and A starts sending packets to B which are acknowledged by B. Complicating the scenario are, however, the following facts. At the beginning of each operating period, a certain amount of time has to be expended by A for sending a preamble for "sync acquisition." Also, at the end of each operating period, the packets that are en route are lost and need retransmission. Finally, the channel is subject to a bit error rate, and packets that are in error also have to be retransmitted.

The purpose of this paper is to provide a queueing model and an algorithmic solution thereof to determine the performance of such meteor burst communication systems. In particular, the model allows for the computation of many important characteristics such as the long run throughput and the statistics for the queue length and sojourn times of messages. Using the algorithm, the effects of the bit error rate and packet lengths on the performance of the system can be evaluated. This paper was inspired by the work of Robert, Mitrani, and King [16] who examined this model by classical queueing techniques and without including some complexities taken into account here.

The paper is organized as follows. In Section II, we provide a general Markov chain model to describe the status of the communication medium in such a way that fairly general distributions can be used to describe the successive periods of availability and unavailability of the medium. In Section III, we then describe a discrete time queueing model for the overall system as a Markov chain. That Markov chain is identified as belonging to the general class of "Markov chains of the $M/G/1$ type," studied extensively by Neuts, Ramaswami, and others. The basic methodology developed for such chains by Neuts [5], [6] and a recently obtained recursion of Ramaswami [14] are also outlined. A key ingredient in that approach is the computation of the minimal nonnegative solution of a nonlinear matrix equation. In Section IV, we show that for the problem at hand, the Markov chain has special properties that allow for substantial simplifications in the computation of this solution and of many related quantities. In Section V, we provide an algorithm to compute the distribution and moments of the sojourn time (response time) of a message of an arbitrary number p of packets. We present some numerical results in Section VI and focus on many interesting questions including the importance of modeling the distribution of the operating periods accurately.

In short, this study provides a tool for analyzing the performance of meteor burst communication systems. The general model and methods presented here have, however, a wider use in that they are applicable in many situations where the communication medium may become unavailable for intermittent periods due to breakdowns or the need to handle higher priority tasks. Although the literature on queueing

Paper approved by the Editor for Computer Network Performance of the IEEE Communications Society. Manuscript received January 30, 1988. The work of M. F. Neuts was supported in part under Grant ECS-8601203 from the National Science Foundation. This paper was invited for presentation at the ORSA/TIMS Conference, St. Louis, MI, April 26, 1988.

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IEEE Log Number 8930074.

systems with service interruptons is extensive (see [3] and its bibliography), exact methods for analyzing them are seldom available. This paper may therefore be of independent interest. For the application which motivated this work, our model is definitely a useful tool, but its effective use requires careful estimation of the distributions of the operating and inoperative periods. Given the interest and existing implementations of such systems, data on these quantities should be available though not readily in the open literature. A particularly useful aspect of our model is that one may use the empirically observed distributions directly as inputs to it.

II. A Model for the Communication Medium

For the discrete time queueing model describing the meteor burst communication system, we take time required to transmit one packet as the time unit. This is determined easily from the known transmission rate and the size of the (fixed size) packets. We further assume that the preamble and the packets lost at the end of a session together constitute M packets. It would appear reasonable to carry out the analysis assuming a fixed value for M and to choose that to be a typical value of a worst case upper bound. For this reason, although our model can be modified easily to handle the situation where the wasted operating period has a random duration, we limit ourselves to the case of a constant M. Finally, we wish to allow for substantially general distributions for the durations of availability and unavailability of the medium without losing computational tractability. This is done as follows.

The state of the communication medium is modeled as a discrete time Markov chain. The state space of that Markov chain is partitioned into two subsets—operating states and inoperative states, so that the medium alternates between operating and inoperative states. The specific model used here assumes that these periods form an alternating renewal process. However, by suitably modifying the Markov chain model, one can allow for correlations between successive operating and inoperative periods without affecting the method of analysis to be developed here. An operating period is further modeled as consisting of two parts—one in which packets from the message queue are sent, and one in which no packets of the message queue are sent. The second one of these is to correspond to the part of the operating period that is wasted due to the preamble and due to those packets at the end of the operating period that always need retransmission. For notational convenience we assume, without loss of generality, that the wasted portions of the operating period are contiguous and occur at the end of each operating period. Later we will show that the assumptions made by us, including the Markovian assumptions, do not affect the generality of the model. Let us now turn to some specifics.

We assume that the states of the communication medium at successive time points are described by an irreducible $m + M + r$ state discrete time Markov chain with stochastic transition matrix of the block partitioned form

$$A = \begin{bmatrix} T & T^0 & 0 & 0 \\ 0 & 0 & I_{M-1} & 0 \\ 0 & 0 & 0 & \beta \\ \bar{b}_M S^0 \alpha & b_M S^0 & S^0 b & S \end{bmatrix}$$

where I_{M-1} is the $(M - 1) \times (M - 1)$ identity matrix, T is an $m \times m$ substochastic matrix, S is an $r \times r$ substochastic matrix, $b = (b_{M-1}, \cdots, b_1)$ is a nonnegative vector of probabilities, $0 \le b_M \le 1$, $0 \le \bar{b}_M = 1 - \Sigma_{j=1}^{M} b_j \le 1$, $T^0 = \mathbf{1} - T\mathbf{1}$ and $S^0 = \mathbf{1} - S\mathbf{1}$ where $\mathbf{1}$ is a column vector (of appropriate order) of 1's, β is an r-component probability vector (i.e., $\beta \ge \mathbf{0}$ and $\beta\mathbf{1} = 1$) and finally α is an m-component probability vector.

The states of the Markov chain are to have the following interpretation. During an interval $[n, n + 1)$, if the Markov chain is in one of the states $m + M + 1, \cdots, m + M + r$, then the communication layer is not present and no packet can be transmitted. If the Markov chain is in $1, \cdots, m + M$, then the communication layer is present and packets may be transmitted. However, if the chain is in the states $m + 1, \cdots, m + M$, then although a packet may be transmitted, it is either part of a preamble or one of those packets occurring at the end of the available period and requiring retransmission; thus in these states also the packet queue cannot be depleted. Thus, the sojourn time in the set of states $m + 1, \cdots, m + M$ corresponds to the operating period wasted. Only when the Markov chain is in one of states $1, \cdots, m$ can one transmit a packet from the true message queue, and such a packet has a probability $\bar{e} = (1 - \epsilon)^b$ of being correctly received where ϵ is the bit error rate and b is the number of bits in a packet.

We will now show that the above Markov chain model is substantially general to support arbitrary distributions for the operating and inoperative periods. In fact, one may even use the empirical distributions obtained from observational data directly.

First of all, note that the durations of unavailable periods are i.i.d. with probability density

$$P(U = k) = \beta S^{k-1} S^0, \quad k \ge 1.$$

Such a density is obtained as the distribution of the absorption time in the discrete time Markov chain with initial probability vector $(\beta, 0)$ and transition matrix $\left[\begin{smallmatrix} S & S^0 \\ 0 & 1 \end{smallmatrix}\right]$ and is called the *phase type distribution PH* (β, S). For a detailed discussion of phase type distributions and their computational uses we refer the reader to [8, ch. 2]. We recall that discrete phase type distributions include as special cases the geometric distribution, the negative binomial distribution, their mixtures and convolutions, as well as any distribution on the nonnegative integers with finite support. Thus, our model allows for very general distributions for the durations of unavailability of the medium. Other authors [12], [16] have assumed that distribution to be geometric (an assumption equivalent to Poisson arrival of meteor scatters). That assumption corresponds to the special case $r = 1$. While it is claimed in the literature that the Poisson assumption for meteor scatters may be justified based on the physics of the underlying phenomena, we are not aware of any validation of this assumption based on actual data. In the general setting considered here, this assumption can, if necessary, be replaced by suitable alternatives.

Just as the distribution of the periods of unavailability is modeled generally, so is the distribution of the periods of availability. Indeed, it is easy to verify that under our general assumptions, the periods of availability are i.i.d. with common density

$$P(V = n) = \begin{cases} b_n & \text{if } n \le M \\ \bar{b}_M \alpha T^{k-1} T^0 & \text{if } n = M + k,\ k \ge 1. \end{cases}$$

Thus, given that the duration of the available period is greater than M, its excess over M has distribution $PH(\alpha, T)$. Because of the stated properties of phase type distributions and the arbitrary nature of b_i, $1 \le i \le M$, we have a general model for the duration of periods of availability as well.

We note that by modifying the elements of the matrix A corresponding to transitions into and out of the states $m + 1, \cdots, m + M$, it is possible to allow the duration of the wasted operating period itself to be a random variable. Clearly, this does not affect the analysis discussed in this paper. Also, since such a specification renders the wasted operating period to have a phase type distribution, once again a general distribution is supported. For reasons stated earlier, we shall in the sequel limit ourselves to the case where M is constant.

The key step in using our model is to choose appropriate distributions for the durations of the operating and inoperative periods. These can then be used to obtain the transition matrix of the Markov chain model. This may be done, for example, by fitting mixtures of negative binomial distribution by a moment matching method. An important fact to note in this context is that our model also allows for the direct use of the empirical distributions obtained from observational data. As noted in the section on numerical examples, the distributions of these periods have a crucial effect on the performance of the system, and therefore careful fitting of these distributions is important for obtaining a valid model for the overall system.

We conclude this section by obtaining intuitively the stability condition for our model. Denoting by $\boldsymbol{\pi}$ the steady-state distribution of the Markov chain describing the status of the communication layer, i.e., $\boldsymbol{\pi} = \boldsymbol{\pi}A$, $\boldsymbol{\pi}\mathbf{1} = 1$, and setting $\boldsymbol{\pi}_1 = (\pi_1, \cdots, \pi_m)$, it follows that the maximum throughput of the system is given by $\bar{e}(\boldsymbol{\pi}_1, \mathbf{1})$. For the system to be stable it is then necessary that the average number of packets $\bar{a}$ arriving per unit time should satisfy $\bar{a} < \bar{e}(\boldsymbol{\pi}_1\ 1)$. This result is proved in the next section.

III. The Queueing Model

Recall that time is discretized in units equal to the transmission time of a single packet. We assume that the number of packets arriving in successive intervals $[n, n + 1)$ are i.i.d. with common distribution $\{a_v : v \geq 0\}$ of mean $\bar{a}$. We define X_n as the number of packets in the system at time $n+$ and J_n, as the state of the medium at time $n-$. The definition of J_n as the state of the medium at $n-$ instead of at $n+$ leads to substantial simplifications in the algorithms. The process $\{(X_n, J_n):n \geq o\}$ is a discrete time Markov chain with state space $\{(i, j):i \geq 0, 1 \leq j \leq m + M + r\}$. Defining the subsets of states $\boldsymbol{i} = \{(i, j):1 \leq j \leq m + M + r\}$ as *level* $\boldsymbol{i}$ and partitioning the transition matrix of the Markov chain according to levels $\mathbf{0}, \mathbf{1}, \cdots$, its transition matrix is given by

$$P = \begin{bmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & A_0 & A_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix} \tag{3.1}$$

where for $v \geq 0$,

$$B_v = a_v A \text{ and } A_v = A\Delta_v \tag{3.2}$$

where Δ_v is the diagonal matrix

$$\Delta_v = \begin{bmatrix} (\bar{e}a_v + ea_{v-1})I_m & 0 \\ 0 & a_{v-1}I_{M+r} \end{bmatrix} \tag{3.3}$$

with I_k denoting the identity matrix of order k. Recall that $\bar{e}$ is the probability that a transmitted packet is error free, and $e = 1 - \bar{e}$. We set $a_{-1} = 0$.

The transition matrix P in (3.1) is a matrix "of the $M/G/1$ type" [5], [11], and is the natural matrix generalization of the structure of the embedded Markov chain of the $M/G/1$ queue (where the blocks reduce to scalars.) Algorithmic methodology for analyzing such chains has been set forth in [5], [6] and has been applied to many examples—see, for example, [7], [9], [13]—and we will draw on those results below. For completeness, we summarize only the key results. No proofs will be repeated; the reader is referred to [5], [11] for the details. We denote by $x(i, j)$ the steady-state probability that the Markov chain $\{(X_n, J_n)\}$ is in state (i, j), and we set $\boldsymbol{x}_i = (x(i, 1), \cdots, x(i, m + M + r))$, $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots)$, and $X(z) = \Sigma_{i=0}^{\infty} z^i \boldsymbol{x}_i$. The key step in the analysis is the computation of the quantities $x(i, j)$ which give the joint stationary distribution of the number of packets in the system and the state of the medium.

1) Stability Condition: By the general results in [5], [11], we know that the Markov chain given by P is ergodic iff $\boldsymbol{\pi}\boldsymbol{\beta}^* < 1$ where, $\boldsymbol{\pi}$ is the stationary probability vector of the transition matrix $\Sigma_{i=0}^{\infty} A_i = A$ and $\boldsymbol{\beta}^* = \Sigma_{i=1}^{\infty} iA_i\mathbf{1}$. A direct computation using (3.2), (3.3) yields that $\boldsymbol{\pi}\boldsymbol{\beta}^* = 1 + \bar{a} - \bar{e}(\boldsymbol{\pi}_1\ \mathbf{1})$ where $\boldsymbol{\pi}_1$ is the vector comprising of the first m components of $\boldsymbol{\pi}$. Therefore, the following holds.

Theorem 1: The queue is stable iff $\bar{a} < \bar{e}(\boldsymbol{\pi}_1\ 1)$.

Henceforth, we only consider stable queues. As already noted, the quanitity $\bar{e}(\boldsymbol{\pi}_1\ \mathbf{1})$ is the maximum achievable throughput, so that the stability condition simply states that the average number of arrivals per time slot should not exceed that quantity.

2) Computation of x_0: The algorithm for the steady-state probability vector $\boldsymbol{x}$ starts by the determination of the vector $\boldsymbol{x}_0$. To that end, we consider the embedded Markov renewal process at visits to the set of states $\mathbf{0}$ whose transition function is given by the generating function $K(z) = \Sigma_{i=0}^{\infty} zB_iG_i(z)$ where $G_i(z)$ is the generating function governing the first passage times from states of $\boldsymbol{i}$ to $\mathbf{0}$. From the structure of P in (3.1), it follows that $G_i(z) = [G_1(z)]^i$, and thus

$$K(z) = \sum_{i=0}^{\infty} zB_i[G(z)]^i \tag{3.4}$$

where $G(z) = G_1(z)$. A familiar result from Markov renewal theory yields $\boldsymbol{x}_0 = (\boldsymbol{\kappa}\boldsymbol{\kappa}^*)^{-1}\boldsymbol{\kappa}$ where $\boldsymbol{\kappa}$ is the stationary probability vector of the Markov chain governed by the transition matrix $K(1)$ and $\boldsymbol{\kappa}^* = K'(1)\mathbf{1}$.

The matrix $G(z) = G_1(z)$ clearly satisfies the equation

$$G(z) = \sum_{i=0}^{\infty} zA_i[G(z)]^i \tag{3.5}$$

and in particular the matrix $G = G(1)$, is known to be the unique stochastic matrix satisfying

$$G = \sum_{i=0}^{\infty} zA_iG^i. \tag{3.6}$$

Successive substitutions in this equation, starting with the zero matrix, give a sequence of matrices monotonically increasing to G and so provide a method to compute that matrix [5]. The special structure of our problem leads to some simplifications in this procedure and, combined with some acceleration techniques recently proposed by Ramaswami [15], this results in an efficient computational scheme to determine G. Having computed G, one evaluates the matrix $K(1)$ and the vector $\boldsymbol{\kappa}$. Direct differentiations in (3.4) and (3.5) lead to the normalization constant $\boldsymbol{\kappa}\boldsymbol{\kappa}^*$ in a computable form. We shall present some of the details of these steps in the next section with particular attention to structural simplifications for the present model.

3) Computation of $\boldsymbol{x}_i$, $i = 1$: Ramaswami [14] has established the following stable recursion for the vectors $\boldsymbol{x}_i$, $i \geq 1$.

Theorem 2: For $i \geq 1$,

$$\boldsymbol{x}_i = [\boldsymbol{x}_0\bar{B}_i + \sum_{j=1}^{i-1} \boldsymbol{x}_j\bar{A}_{i+1-j}](I - \bar{A}_1)^{-1} \tag{3.7}$$

where $\bar{B}_i = \Sigma_{j=i}^{\infty} B_jG^{j-i}$ and $\bar{A}_i = \Sigma_{j=i}^{\infty} A_jG^{j-i}$.

The matrices $\bar{A}_i$ and $\bar{B}_i$ obviously tend to the zero matrix as $i \to \infty$. One may therefore choose a sufficiently large index i, set $\bar{A}_i = \bar{B}_i = 0$, and compute the others using the backward recursions $\bar{A}_k = A_k + \bar{A}_{k+1}G$ and $\bar{B}_k = B_k + \bar{B}_{k+1}G$ and substitute these in (3.7) to evaluate the vectors $\boldsymbol{x}_i$, $i \geq 1$. Particularly useful in confirming the adequacy of the truncation index is the obvious result

$$X(1) = \sum_{i=0}^{\infty} \boldsymbol{x}_i = \boldsymbol{\pi} \tag{3.8}$$

which follows from consideration of the stationary distribution of the state of the communication medium.

Modulo the details of computing the matrix G and the vector x_0, the preceding results provide the relevant algorithms for the stationary probabilities $x(i, j)$. Once these are computed, one can routinely derive from them the joint stationary distributions of the system size and the state of the medium at special epochs such as the beginning of an operating period, the end of an operating period, etc. These provide useful information about how the queue builds up during inoperative periods and becomes depleted in the operating periods.

The steady state equation $\boldsymbol{x} = \boldsymbol{x}P$ may be written as

$$\boldsymbol{x}_i = \boldsymbol{x}_0 B_i + \sum_{j=1}^{i+1} \boldsymbol{x}_j A_{i+1-j}, \quad i \geq 0. \tag{3.9}$$

The generating function $X(z)$ clearly satisfies

$$X(z)[zI - A(z)] = \boldsymbol{x}_0[zB(z) - A(z)] \tag{3.10}$$

where $A(z) = \sum_{i=0}^{\infty} z^i A_i$ and $B(z) = \sum_{i=0}^{\infty} z^i B_i$. Repeatedly differentiating in (3.10) and using standard techniques (see [11]), one can compute the moments of the system size. These provide further checks on the accuracy of the computed stationary probabilities. We omit the details and discuss in Section IV, the simplifications in these computations resulting from the special structure of P for the present problem.

IV. Simplifications Due To Structure

From (3.2), (3.3) the last $M + r$ columns of the matrix A_0 are zero. This implies that the last $M + r$ columns of the matrix G are zero; each of the iterates in the successive substitution scheme in (3.6) has this property. The matrix G therefore is of the form

$$G = \begin{bmatrix} G_1 & 0 \\ G_2 & 0 \end{bmatrix} \tag{4.1}$$

where G_1 is of order m, and G_2 of order $(M + r) \times m$. Subsituting this in (3.6) and similarly partitioning A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is of order m and A_{21} is of order $(M + r) \times m$, and using (3.2), (3.3), we see that the matrices G_1 and G_2 satisfy the equations

$$G_1 = \bar{e}A_{11}H + eA_{11}HG_1 + A_{12}G_2H \tag{4.2}$$

$$G_2 = \bar{e}A_{21}H + eA_{21}HG_1 + A_{22}G_2H \tag{4.3}$$

where

$$H = \sum_{j=0}^{\infty} a_j G_1^j. \tag{4.4}$$

Successive substitutions in these equations result in the scheme.

Initialization: Set $G_1(1) = 0_{m\times m}$ and $G_2(1) = 0_{(m+r)\times m}$.

Iteration: For $n \geq 1$, set

$$H(n+1) = \sum_{j=0}^{\infty} a_j [G_1(n)]^j$$

$$G_1(n+1) = \bar{e}A_{11}H(n+1) + eA_{11}H(n+1)G_1(n) + A_{12}G_2(n)H(n+1)$$

$$G_2(n+1) = \bar{e}A_{21}H(n+1) + eA_{21}H(n+1)G_1(n) + A_{22}G_2(n)H(n+1).$$

It follows that as $n \uparrow \infty$, $G_1(n) \uparrow G_1$, and $G_2(n) \uparrow G_2$. This reduces the number of matrix multiplications substantially and avoids unnecessary multiplications by zero.

While as shown in [15], this scheme converges, its convergence is usually slow. An accelerated scheme for computing G_1 and G_2 adapts the sub-Newton scheme proposed in [15]. This corresponds to the following technique. Having computed in $(n + 1)$st iterate, say $G(n + 1)$ of G using the above scheme, update $G(n + 1)$ by setting

$$G(n+1) \leftarrow G(n+1) + A_1 Z_{n+1} + A_2[Z_{n+1}G(n) + G(n)Z_{n+1}]$$

where

$$Z_{n+1} = (I - A_1)^{-1}[G(n+1) - G(n)].$$

In [15] it is shown that this modification also yields a monotonically converging sequence of iterates, and extensive experimentation has shown this scheme to be substantially faster than the direct scheme. In practice, the above acceleration is implemented in the appropriate partitioned form for the iterates $G_1(n)$ and $G_2(n)$.

The iterations for G are continued until the maximum difference in the corresponding elements of two successive iterates falls below a specified tolerance (taken by us to be 10^{-8}). Also, at the termination of the iterative scheme, a linear extrapolation to a stochastic matrix based on the last two iterates is computed and taken as the computed value of the matrix G. This summarizes the steps in computing the matrix G.

The structure of G also simplifies the computation of the matrix $K(1)$. From (3.4), we have that

$$K(1) = A \begin{bmatrix} \sum_{k=0}^{\infty} a_k G_1^k & 0 \\ G_2 \sum_{k=1}^{\infty} a_k G_1^{k-1} & a_0 I_{M+r} \end{bmatrix}.$$

Computation of $K(1)$ from this formula results in considerable savings. We recall that the steady-state vector $\boldsymbol{x}_0$ is given by $\boldsymbol{x}_0 = (\boldsymbol{\kappa}\boldsymbol{\kappa}^*)^{-1}\boldsymbol{\kappa}$ where $\boldsymbol{\kappa}$ is the invariant probability vector of $K(1)$ and $\boldsymbol{\kappa}^* = K'(1)\,\mathbf{1}$. The vector $\boldsymbol{\kappa}$ is evaluated using any of the standard technique for computing the stationary probability vector of a finite state Markov chain; we used the algorithm developed in [4]. After some routine algebra, details of which are omitted, it is possible to show that for our model

$$(\boldsymbol{\kappa}\boldsymbol{\kappa}^*) = (1-\rho)^{-1}\left[1 - \boldsymbol{\kappa}\begin{bmatrix} A_{12}\mathbf{1} \\ A_{22}\mathbf{1} \end{bmatrix}\right]$$

where $\rho = \boldsymbol{\pi}\boldsymbol{\beta}^* = 1 + \bar{a} - (\boldsymbol{\pi}_1\ \mathbf{1})\bar{e}$. In short, the determination of $\boldsymbol{x}_0$ is fairly straightforward once the matrix G has been computed.

V. Response Time of a Message

Of particular interest to the meteor burst communication system is the response (sojourn) time distribution and its moment of a message of a given length p under the FIFO discipline. The queue is assumed to be in steady state at time 0, and we consider the response (sojourn) time of a message of length p arriving at time 0. Since the queue is FIFO, we may study its distribution by proceeding as though arrivals were shut off in the interval $(0, \infty)$.

Let

$$C_0 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}$$

and let $C_1 = A - C_0$. Given that the arrival at time 0 finds k packets ahead of it and that the medium is in state i, the response time of that message is the first passage time from the state $(k + p, i)$ to level $\mathbf{0}$ in the Markov chain with transition matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots \\ C_0 & C_1 & 0 & 0 & \cdots \\ 0 & C_0 & C_1 & 0 & \cdots \\ 0 & 0 & C_0 & C_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{bmatrix}. \tag{5.1}$$

This follows since a packet is successfully transmitted iff the Markov chain describing the medium enters the set $\{1, \cdots, m\}$ and the packet has no error. The stationary probability of state (k, i) is $x(k, i)$, so that the probability generating function of the response time distribution of a message of size p is given by

$$W_p(z) = \sum_{k=0}^{\infty} \mathbf{x}_k [H(z)]^{k+p} \mathbf{1} \tag{5.2}$$

where

$$H(z) = [1 - z(A - C_0)]^{-1} z C_0 \tag{5.3}$$

is the probability generating function of the first passage time from any level $\mathbf{i}$ to $\mathbf{i} - \mathbf{1}$.

As in other models of $M/G/1$ type—see, for example, [10], [13]—it is possible by using the equations (3.9), to obtain a matrix generalization of the Pollaczek-Khinchin formula for $W_p(z)$. The resulting expressions appear, however, to be too complicated for computational use. We shall not present that result, and only report on the direct scheme which we have implemented to compute the quantities of interst.

Sojourn Time Distribution: For $j, k \geq 0$, let the ith component of the $(m + M + r)$ column vector $\mathbf{r}_k^{(j)}$ be the probability that, starting with an unlimited queue of packets to draw from and the medium in state i, the system successfully transmits at most k packets during the first j time slots. Then, clearly

$$\mathbf{r}_k^{(j)} = \mathbf{1}, \text{ for all } k \geq j \tag{5.4}$$

and further

$$\mathbf{r}_k^{(j)} = C_0 \mathbf{r}_{k-1}^{(j-1)} + C_1 \mathbf{r}_k^{(j-1)}, \text{ for all } 0 \leq k \leq j, \text{ and } j \geq 1. \tag{5.5}$$

Now, the probability $w_p(j)$ that the sojourn time of a message of size p exceeds j is given by

$$w_p(j) = \sum_{k=0}^{\infty} \mathbf{x}_k \mathbf{r}_{k+p-1}^{(j)}, \tag{5.6}$$

since the arriving message of size p spends more than j units of time in the system iff during the first j time units at most $k + p - 1$ packets are succesfully transmitted where k is the number of packets ahead of the arrival. Computation of the complementary sojourn time distribution using the formulae is fairly straightforward.

One may differentiate in (5.2) and (5.3) to obtain the factorial moments $W_p'(1)$ and $W_p''(1)$ of the sojourn time distribution in computable forms from which the mean and the variance of the sojourn time readily follow. These moment formulae provide accuracy checks in the computation of the sojourn time distribution. As these are now routine algorithmic steps, we omit the details.

VI. Numerical Examples

Here we report the results of computations for a selected set of examples. In all of them we have used the same set of values for the means of the operating period and the inoperative period as well as for the arrival rate. However, it will be seen that depending on the particular set of distributions used, the performance characteristics differ significantly. Thus, the principle conclusion to be drawn from the examples is that the distributions of the operating and the inoperative periods have a significant effect on performance. In particular, the knowledge of their means alone is not adequate to predict accurately the performance of the system. For proper use of the model and algorithms, it is essential that accurate distributional assumptions be made. As our model allows the use of general distributions (and, in particular, the empirical distributions themselves), the techniques presented here can be fruitfully employed to study the performance of meteor scatter systems under realistic assumptions. Finally, in contrast to the distributional assumptions, the bit error rate and the number of packets lost during an operating period (due to sync acquisition time, etc.) do not appear to have a significant effect on performance.

TABLE I
SELECTED STATISTICS FOR SYSTEM SIZE (N) AND SOJOURN TIME (S)

Inoperative Period		Operative Period		
		Neg. Bin.	Geometric	Mixture
Neg. Bin.	$P(N=0)$	0.74	0.62	0.53
	$E(N)$	5.97	9.75	13.79
	$\sigma(N)$	11.60	16.09	20.70
	$E(S)$	553.20	789.14	1091.02
	$\sigma(S)$	405.91	732.61	1133.13
Geometric	$P(N=0)$	0.63	0.54	0.47
	$E(N)$	10.83	14.98	19.30
	$\sigma(N)$	18.29	22.83	27.52
	$E(S)$	1021.66	1311.41	1651.61
	$\sigma(S)$	1064.92	1398.71	1800.75
Mixture	$P(N=0)$	0.53	0.47	0.42
	$E(N)$	18.02	22.50	27.14
	$\sigma(N)$	27.38	32.09	36.96
	$E(S)$	1721.02	2063.47	2445.31
	$\sigma(S)$	1929.30	2304.63	2731.48

Our examples do not pertain to any specific system and are based on some hypothetical cases, but as far as possible, we have chosen parameter values representative of actual realizations. The time to transmit a packet is 10 ms. This is also the unit of time. Arrivals occur in messages of 20 packets each with a mean arrival rate of 0.01 packets per unit time. This is modeled by assuming that the distribution $\{a_j\}$ is given by $a_0 = 0.9995$ and $a_0 = 0.0005$. Each packet consists of 20 bits. We set the mean operating period to 58 units (0.58 s) and the mean inoperative period to 1000 units (10 s); as reported in [12], these values are typical of some systems for which data are available.

The first set of Tables I–V, demonstrate the effect of the distributional assumptions on performance characteristics. Throughout, $M = 4$ and the bit error rate $\epsilon = 10^{-4}$. The tables correspond to three distributions each for the operating period and for the inoperative period. Those used for the operating period are the geometric distribution with mean 58, the negative binomial distribution with 9 phases and mean 58, and an equally weighted mixture of two geometric disributions with means 100 and 16, respectively. Similarly, the distributions considered for the inoperative period are the geometric distribution with mean 1000, the negative binomial distribution with 9 phases and mean 1000, and the equally weighted mixture of two geometric distributions with means 200 and 1800, respectively. In each case, the negative binomial distribution has less variability than the geometric which in turn has less variability than the mixture.

Table I lists the following steady-state measures for the resulting systems: the probability of emptiness ($P(N = 0)$), the expected number of packets in the system ($E(N)$), the *s.d.* of the number of packets in the system $\sigma(N)$, the mean sojourn time ($E(S)$) of a packet, and the *s.d.*($\sigma(S)$) of the sojourn time of a packet. At a glance this table shows that the distributional assumptions have a drastic effect on the computed perform-

TABLE II
SELECTED PERCENTILES OF SYSTEM SIZE

Inoperative Period	Percentile	Operative Period: Neg. Bin.	Geometric	Mixture
Neg. Bin.	50	0	0	0
	75	5	20	20
	90	20	32	40
	95	26	40	57
Geometric	50	0	0	7
	75	20	20	28
	90	40	40	58
	95	40	60	77
Mixture	50	0	13	20
	75	20	40	40
	90	60	60	79
	95	80	85	100

TABLE III
SELECTED PERCENTILES OF SOJOURN TIME

Inoperative Period	Percentile	Operative Period: Neg. Bin.	Geometric	Mixture
Neg. Bin.	50	498	619	759
	75	806	1055	1430
	90	1102	1675	2497
	95	1294	2206	3359
Geometric	50	693	874	1077
	75	1431	1823	2281
	90	2407	3107	3948
	95	3147	4095	5248
Mixture	50	1085	1313	1560
	75	2439	2911	3434
	90	4235	5047	5966
	95	5595	6678	7912

TABLE IV
PERCENTILES OF THE SYSTEM SIZE AT THE BEGINNING OF AN INOPERATIVE PERIOD

Inoperative Period	Percentile	Operative Period: Neg. Bin.	Geometric	Mixture
Neg. Bin.	50	0	0	0
	75	0	0	12
	90	0	19	32
	95	0	30	46
Geometric	50	0	0	0
	75	0	0	11
	90	0	20	37
	95	6	37	56
Mixture	50	0	0	0
	75	0	0	11
	90	0	23	41
	95	15	47	69

TABLE V
PERCENTILES OF THE SYSTEM SIZE AT THE END OF AN INOPERATIVE PERIOD

Inoperative Period	Percentile	Operative Period: Neg. Bin.	Geometric	Mixture
Neg. Bin.	50	0	5	19
	75	20	20	27
	90	35	40	48
	95	40	52	61
Geometric	50	0	0	11
	75	20	20	31
	90	40	40	60
	95	42	60	78
Mixture	50	0	0	0
	75	20	20	33
	90	40	57	63
	95	60	80	93

ance measures. A cycle comprising of a long inoperative period and a short operating period leaves behind a substantial backlog, which usually persists for a significant amount of time. This inflates the overall means for the system size and the sojourn time. This further results in increased variability in the distributions of these quantities. For the distributions considered, note that such effects can be expected to be least pronounced for the negative binomial and most for the mixture.

Tables II–V confirm these findings through more detailed information on the queue. In Tables II and III, we give

TABLE VI
MAXIMUM THROUGHPUT

	M=1	M=4	M=7
$\epsilon=10^{-4}$	0.0538	0.0510	0.0484
$\epsilon=10^{-3}$	0.0528	0.0501	0.0476
$\epsilon=10^{-2}$	0.0441	0.0418	0.0397

TABLE VII
SELECTED STATISTICS FOR SYSTEM SIZE (N) AND SOJOURN TIME (S)

		$M=1$	$M=4$	$M=7$
$\epsilon=10^{-4}$	$P(N=0)$	0.558	0.541	0.524
	$E(N)$	14.01	14.98	16.03
	$\sigma(N)$	21.76	22.83	22.97
	$E(S)$	1223.84	1311.41	1406.51
	$\sigma(S)$	1307.53	1398.71	1497.82
$\epsilon=10^{-3}$	$P(N=0)$	0.555	0.539	0.522
	$E(N)$	14.11	15.09	16.16
	$\sigma(N)$	21.87	22.94	24.10
	$E(S)$	1230.82	1319.34	1415.57
	$\sigma(S)$	1315.66	1407.93	1508.33
$\epsilon=10^{-2}$	$P(N=0)$	0.529	0.512	0.494
	$E(N)$	15.31	16.42	17.64
	$\sigma(N)$	23.11	24.32	25.63
	$E(S)$	1314.59	1414.73	1524.38
	$\sigma(S)$	1413.72	1519.20	1634.77

selected percentiles of the number of packets in the system and of the sojourn time of a packet. Note the pronounced tails when the input distributions exhibit greater variability. Tables IV and V, respectively, give the conditional distributions of the number in the system at the beginning and at the end of an inoperative period. When both distributions are negative binomial, we see that almost all the packets arriving during an inoperative period are cleared from the system during the succeeding operating period. However, when the distributions are not so regular, a significant queue can build up. For example, when both distributions are mixtures, then with significant probability there is a cycle with a rather long inoperative and a short operating period. Such cycles leave behind a long queue which takes a long time to dissipate.

The results in Tables I–V show that performance predictions based only on mean values can be widely off the mark. In practice, typical performance criteria require that, with high probability, the response time should fall below stipulated values. These are conditions on the tails of distributions which are highly sensitive to the distributional assumptions made on the inputs. This shows the importance of gathering detailed distributional information on the operating and inoperative periods in making correct performance predictions.

In Tables VI and VII, we examine the effects of the values of M and ϵ on the performance of the system. These tables are computed assuming that both the operating and the inoperative period have geometric distributions. One sees that in the range considered, these parameters exert little influence on the performance characteristics.

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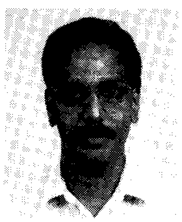

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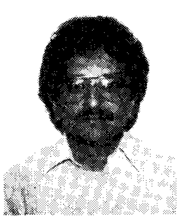

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