MA 211 Numerical Methods for PDEs

Numerical Methods for Hyperbolic PDEs

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1. Show that the solution of the difference scheme (*Lax-Friedrichs scheme*):
$$u_m^{n+1} = \frac{1}{2}(u_{m+1}^n + u_{m-1}^n) - \frac{R}{2}(u_{m+1}^n - u_{m-1}^n)$$
converges to the 1D hyperbolic PDE $u_t + au_x = 0$, for $|R| \leq 1$, $R = ak/h$.

2. Show that the following difference scheme is a $O(k + h^4)$ approximation of $u_t + au_x = 0$:
$$u_m^{n+1} = u_m^n - \frac{R}{2}\delta_0 u_m^n + \frac{R}{12}\delta_0 u_m^n + \frac{R^2}{2}\left(\frac{4}{3} + R^2\right)(u_{m-1}^n - 2u_m^n + u_{m+1}^n) - \frac{R^2}{8}\left(\frac{1}{3} + R^2\right)\delta_0^2 u_m^n, \; m = 1, 2, \ldots,$$
where $\delta_0^2 = \delta_0\delta_0$, and $\delta_0 u_m^n = u_{m+1}^n - u_{m-1}^n$. Discuss the assumptions that must be made on the derivatives of the solution to the PDE that are necessary to make the above statement true.

3. Determine the order of accuracy of the following difference schemes to the first-order hyperbolic PDE $u_t + au_x = 0$:

   (a) *Leapfrog scheme:* $u_m^{n+1} = u_m^{n-1} - R(u_{m+1}^n - u_{m-1}^n)$.

   (b) $u_m^{n+1} = u_m^{n-1} - R(u_{m+1}^n - u_{m-1}^n) + \dfrac{R}{6}(u_{m+1}^n - u_{m-1}^n)$.

   (c) $u_m^{n+2} = u_m^{n-2} - \dfrac{2R}{3}\left(1 - \dfrac{1}{6}\delta^2\right)\delta_0(2u_m^{n+1} - u_m^n + 2u_m^{n-1})$.

   (d) $u_m^{n+1} = u_m^{n-1} - R\delta_0 u_m^n + \dfrac{R}{6}\delta^2\delta_0 u_m^n - \dfrac{R}{30}\delta^4\delta_0 u_m^n$, where $\delta^4 = \delta^2\delta^2$, and $\delta^2 u_m^n = (u_{m-1}^n - 2u_m^n + u_{m+1}^n)$.

   Further, discuss about the stability condition for all these schemes via. Von-Neumann stability analysis.

4. Find the numerical domain of dependence and the CFL condition for the difference schemes given in **Problem** 3.

5. Solve the following 1D hyperbolic PDE by the forward explicit and implicit schemes (FTFS and BTFS):
$$\begin{cases} u_t - 2u_x = 0, \quad x \in (0,1), \, t > 0 \\ u(x,0) = 1 + \sin(2\pi\, x), \quad x \in [0,1] \\ u(1,t) = 1.0 \end{cases}$$
Take $\Delta\, x = h = 0.25$, and $\Delta\, t = k = 0.001$. Determine the solution for three time-levels.

6. Solve the following 1D hyperbolic PDE by the backward explicit and implicit schemes (FTBS and BTBS):

$$\begin{cases} u_t + 2u_x = 0, & x \in (0,1),\, t > 0 \\ u(x,0) = 1 + \sin(2\pi\, x), & x \in [0,1] \\ u(0,t) = 1.0 \end{cases}$$

Take $\Delta\, x = h = 0.25$, and $\Delta\, t = k = 0.001$. Determine the solution for three time-levels.

7. Solve the 1D hyperbolic PDEs given in **Problem** 5 by *Lax-Wendroff scheme*. Use the following numerical boundary conditions at $x = 0$:

   (i) $u_0^n = 1.0.$

   (ii) $u_0^n = u_1^n.$

   (iii) $u_0^{n+1} = u_0^n - \frac{a\Delta\, t}{\Delta\, x}(u_1^n - u_0^n).$

   (iv) $u_0^{n+1} + u_1^{n+1} + \frac{a\Delta\, t}{\Delta\, x}(u_1^{n+1} - u_0^{n+1}) = u_0^n + u_1^n + \frac{a\Delta\, t}{\Delta\, x}(u_1^n - u_0^n)$

8. Solve the first-order hyperbolic PDEs given in **Problem** 5 by *Crank-Nicolson scheme* with the numerical boundary conditions given in **Problem** 7.

9. Consider the following 1D hyperbolic initial-boundary-value problem:

$$\begin{cases} u_t - u_x = 0, & x \in (0,1),\, t > 0 \\ u(x,0) = f(x), & x \in [0,1] \\ u(0,t) = u(1,t), \end{cases}$$

where the initial-value function f is defined as

$$f(x) = \begin{cases} 1, & \text{if } 0.4 \leq x \leq 0.6 \\ 0, & \text{otherwise} \end{cases}$$

Obtain the numerical solution of this initial-boundary-value problem with $\Delta\, x = h = 0.25$, and $\Delta\, t = k = 0.001$, for three time levels, by the following schemes:

   (i) FTFS scheme.

   (ii) Lax-Wendroff scheme.

   (iii) Lax-Friedrichs scheme.

   (iv) Crank-Nicolson scheme.

10. Solve the IBVP given **Problem 9** by taking the initial function f as $f(x) = \sin^2(\pi\, x)$.