Best Uniform Approximation to Continuous Functions from Finite Dimensional Spaces of Continuous Functions - from 1855 to 2013

Aldric L. Brown
Department of Mathematics
University College London

In the 1850's P. L. Chebyshev considered best uniform approximations to a function $f : [0,1] \to \mathbb{R}$ from the space of real polynomial functions of degree $\leq n-1$. He discovered the phenomenon of uniqueness of best approximations and characterised them - Approximation Theory was born.

Now one considers a more general situation. Let T be a compact Hausdorff space (for example an interval, a circle, a square); let $C(T)$ be the space of real continuos functions $f : T \to \mathbb{R}$, equipped with the uniform norm

$$\|f\| = \max\{|f(t)| : t \in T\};$$

let M be a finite dimensional subspace of $C(T)$. If f is in $C(T)$ then we define the distance

$$\operatorname{dist}(f, M) = \min\{\|f-g\| : g \in M$$

and we let

$$P_M(f) = \{g \in M : \|f-g\| = \operatorname{dist}(f, M)\}$$

denote the set of best uniform approximations to f from M. So $P_M : C(T) \to \{W : W \subseteq M\}$ is a *set-valued mapping*; it is called the *metric projection* of $C(T)$ onto M. For each $f \in C(T)$, $P_M(f)$ is a non-empty compact convex subset of M.

In the 1950's there arose (naturally, from a result in the theory of integral equations) the question whether, given M there exists a continuous selection for P_M; that is, a continuous $s : C(T) \to M$ such that $s(f) \in P_M(f)$ for all $f \in C(T)$. If P_M is lower semi-continuous then the answer is 'Yes' (Michael, 1956). Since then there has been a steady development of a theory of these metric projections. The talk will describe how recent results (2013) have changed the story, brought it closer to its origins, and generated hope for the unanswered questions.

A. L. Brown
09-09-14