FDTD: An Introduction

- PML could be analyzed in stretched coordinate systems
- Source-free Maxwell's equations

$$\nabla_{stretched} \times \vec{E} = -j\omega\mu\vec{H}; \nabla_{stretched} \times \vec{H} = j\omega\varepsilon\vec{E};$$

$$\nabla_{stretched} \bullet \left(\varepsilon\vec{E}\right) = 0; \nabla_{stretched} \bullet \left(\mu\vec{H}\right) = 0;$$

$$\nabla_{stretched} = \hat{x}\frac{1}{s_x(x)}\frac{\partial}{\partial x} + \hat{y}\frac{1}{s_y(y)}\frac{\partial}{\partial y} + \hat{z}\frac{1}{s_z(z)}\frac{\partial}{\partial z}$$

- Using this new definition of del operator, we can find components for the two Maxwell's curl equations

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For example

- x-component of first Maxwell's curl equation is

$$\frac{1}{s_y}\frac{\partial H_z}{\partial y}-\frac{1}{s_z}\frac{\partial H_y}{\partial z}=j\omega\varepsilon E_x$$

- In time domain, multiplication becomes convolution

$$\bar{S}_y*\frac{\partial H_z}{\partial y}-\bar{S}_z*\frac{\partial H_y}{\partial z}=\varepsilon\frac{\partial E_x}{\partial t};\bar{S}_y=F^{-1}\left(\frac{1}{s_y}\right),\bar{S}_z=F^{-1}\left(\frac{1}{s_z}\right)$$

- * means convolution, that's why convolutional PML, supposedly the best PML

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- Note

$$\frac{1}{s_x}=\frac{1}{k_x+\dfrac{\sigma_x}{\alpha_x+j\omega\varepsilon}}=\frac{\alpha_x+j\omega\varepsilon}{\alpha_x k_x+j\omega\varepsilon k_x+\sigma_x}$$

- which can be expressed as

$$\frac{1}{s_x}=\frac{1}{k_x}-\frac{1}{B_x};\frac{1}{B_x}=\frac{\sigma_x}{k_x}\left(\frac{1}{\alpha_x k_x+j\omega\varepsilon k_x+\sigma_x}\right)$$

$$\frac{1}{s_i}=\frac{1}{k_i}-\frac{1}{B_i};\frac{1}{B_i}=\frac{\sigma_i}{k_i}\left(\frac{1}{\alpha_i k_i+j\omega\varepsilon k_i+\sigma_i}\right),i\in\{x,y,z\}$$

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- Therefore, inverse FT of inverse of scale factor s_i

$$\bar{S}_i = F^{-1}\left(\frac{1}{s_i}\right) = F^{-1}\left(\frac{1}{k_i} - \frac{\sigma_i}{k_i}\left(\frac{1}{\alpha_i k_i + j\omega\varepsilon k_i + \sigma_i}\right)\right)$$

$$= F^{-1}\left(\frac{1}{k_i} - \frac{\sigma_i}{k_i \varepsilon k_i}\left(\frac{1}{\frac{\alpha_i}{\varepsilon} + j\omega + \frac{\sigma_i}{\varepsilon k_i}}\right)\right)$$

$$= \frac{\delta(t)}{k_i} - \frac{\sigma_i}{k_i^2 \varepsilon} \exp\left(-\left(\frac{\alpha_i}{\varepsilon} + \frac{\sigma_i}{k_i \varepsilon}\right)t\right) = \frac{\delta(t)}{k_i} + Q_i(t)$$

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- Hence,

$$\frac{1}{k_y}\frac{\partial H_z}{\partial y}-\frac{1}{k_z}\frac{\partial H_y}{\partial z}+Q_y(t)*\frac{\partial H_z}{\partial y}-Q_z(t)*\frac{\partial H_y}{\partial z}=\varepsilon\frac{\partial E_x}{\partial t}$$

- Define a new Ψ function for the 3rd and 4th convolution terms in the above equation

$$\Psi^n_{E_j w}=Q_w(t)*\frac{\partial H_k}{\partial w};t=n\Delta t,E_j\in(E_x,E_y,E_z),w,k\in(x,y,z);j\neq w\neq k$$

- Notation for $\Psi^n_{E_j w}$
  - Subscript of Ψ reveals that this function appears in the update of E_j
  - Subscript w shows that spatial derivative is with respect to w
  - Superscript n is the time running index

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$$\Psi^n_{E_j w} = Q_w(t) * \frac{\partial H_k}{\partial w}; t = n\Delta t, E_j \in (E_x, E_y, E_z), w, k \in (x, y, z); j \neq w \neq k$$

- Note that in this integration

$$\Psi^n_{E_j w} = \int_{\tau=0}^{n\Delta t} Q_w(\tau) \frac{\partial H_k(n\Delta t - \tau)}{\partial w} d\tau$$

- The integrand is zero for τ<0 since $Q_w(\tau) = 0, \tau < 0$
- and τ=0 is lower limit of integration
- Also the upper limit on the integration τ=nΔt can be observed from

$$H_k(n\Delta t - \tau) = 0; n\Delta t - \tau < 0 \Rightarrow H_k(n\Delta t - \tau) = 0; n\Delta t < \tau$$

- Second function and integrand is zero for $n\Delta t < \tau$

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- In FDTD, H_k varies in discrete time

$$\frac{\partial H_k(t)}{\partial w} = \sum_{j=o}^{J_{\max}} \frac{\partial H_k(n\Delta t)}{\partial w} p_j(t); p_j(t) = \begin{cases} 1, & j\Delta t \le t < (j+1)\Delta t \\ 0, & otherwise \end{cases}$$

To find $\dfrac{\partial H_k(n\Delta t - \tau)}{\partial w}$

Note that $H_k(n\Delta t - \tau)$ is basically flipping and shifting

Hence

$$f(n\Delta t - \tau) = \sum_{j=0}^{n-1} f_{n-j} p_j(\tau)$$

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$$\Rightarrow \frac{\partial H_k\left(n\Delta t-\tau\right)}{\partial w}=\sum_{j=o}^{n-1}\frac{\partial H_k^{n-j}}{\partial w}p_j\left(\tau\right)$$

At time step n

$$\Psi_{E_j w}^{n}=\int_{\tau=0}^{n\Delta t}Q_w\left(\tau\right)\sum_{j=o}^{n-1}\frac{\partial H_k^{n-j}}{\partial w}p_j\left(\tau\right)d\tau$$

Interchanging the summation and integration

$$\Psi_{E_j w}^{n}=\sum_{j=o}^{n-1}\frac{\partial H_k^{n-j}}{\partial w}\int_{\tau=0}^{n\Delta t}Q_w\left(\tau\right)p_j\left(\tau\right)d\tau$$

Pulse function is equal to 1 for the interval $j\Delta t\le t<\left(j+1\right)\Delta t$

$$\Psi_{E_j w}^{n}=\sum_{j=o}^{n-1}\frac{\partial H_k^{n-j}}{\partial w}\int_{\tau=j\Delta t}^{\tau=\left(j+1\right)\Delta t}Q_w\left(\tau\right)d\tau$$

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Since Q_w is an exponential function, we can easily find it's integration as

$$\int_{\tau=j\Delta t}^{\tau=(j+1)\Delta t} Q_w(\tau)\,d\tau = \int_{\tau=j\Delta t}^{\tau=(j+1)\Delta t} \frac{\sigma_i}{k_i^2\varepsilon}\exp\left(-\left(\frac{\alpha_i}{\varepsilon}+\frac{\sigma_i}{k_i\varepsilon}\right)t\right)d\tau = C_w\left(b_w\right)^j$$

- where

$$b_w = \exp\left(-\left(\frac{\alpha_i}{\varepsilon}+\frac{\sigma_i}{k_i\varepsilon}\right)\Delta t\right); C_w = \frac{\sigma_w}{\sigma_w k_w + \left(k_w\right)^2\alpha_w}\left(b_w - 1\right)$$

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- Now $$\Psi^{n}_{E_j w} = \sum_{j=o}^{n-1} \frac{\partial H_k^{n-j}}{\partial w} C_w \left(b_w\right)^j$$

- Separating j=0 and j=1,…,n-1 terms, we have,

$$\Psi^{n}_{E_j w} = C_w \frac{\partial H_k^{n}}{\partial w} + \sum_{j=1}^{n-1} \frac{\partial H_k^{n-j}}{\partial w} C_w \left(b_w\right)^j$$

- Putting j=l+1

$$\Psi^{n}_{E_j w} = C_w \frac{\partial H_k^{n}}{\partial w} + \sum_{l=0}^{n-2} \frac{\partial H_k^{n-l-1}}{\partial w} C_w \left(b_w\right)^{l+1}$$

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- Taking out b_w as a common factor in the second term

$$\Psi^{n}_{E_j w} = C_w \frac{\partial H^{n}_{k}}{\partial w} + b_w \left\{ \sum_{l=0}^{(n-1)-1} \frac{\partial H^{(n-1)-l}_{k}}{\partial w} C_w \left(b_w\right)^{l} \right\}$$

- Observe that the term in second bracket in the above expression is exactly $\Psi^{n-1}_{E_j w}$
- Hence, the update equation for

$$\Psi^{n}_{E_j w} = C_w \frac{\partial H^{n}_{k}}{\partial w} + b_w \Psi^{n-1}_{E_j w}$$

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$$\Psi^{n}_{E_j w} = C_w \frac{\partial H^n_k}{\partial w} + b_w \Psi^{n-1}_{E_j w}$$

- Discretization of CPML

$$\varepsilon \frac{\partial E_x}{\partial t} = \frac{1}{k_y}\frac{\partial H_z}{\partial y} - \frac{1}{k_z}\frac{\partial H_y}{\partial z} + \boxed{Q_y(t) * \frac{\partial H_z}{\partial y}} - \boxed{Q_z(t) * \frac{\partial H_y}{\partial z}}$$

$$\varepsilon \frac{E_x\Big|^{n+1}_{i+1/2,j,k} - E_x\Big|^{n}_{i+1/2,j,k}}{\Delta t}$$

$$= \frac{H_z\Big|^{n+1/2}_{i+1/2,j+1/2,k} - H_z\Big|^{n+1/2}_{i+1/2,j-1/2,k}}{k_y \Delta y} - \frac{H_y\Big|^{n+1/2}_{i+1/2,j,k+1/2} - H_y\Big|^{n+1/2}_{i+1/2,j,k-1/2}}{k_z \Delta z}$$

$$+ \Psi_{E_x,y}\Big|^{n+1/2}_{i+1/2,j,k} - \Psi_{E_x,z}\Big|^{n+1/2}_{i+1/2,j,k}$$

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- and

$$\Psi^{n}_{E_j w} = b_w \Psi^{n-1}_{E_j w} + C_w \frac{\partial H^n_k}{\partial w}$$

$$\Psi_{E_x,y}\Big|^{n+1/2}_{i+1/2,j,k} = b_y \Psi_{E_x,y}\Big|^{n-1/2}_{i+1/2,j,k} + C_y \frac{H_z\Big|^{n+1/2}_{i+1/2,j+1/2,k} - H_z\Big|^{n+1/2}_{i+1/2,j-1/2,k}}{\Delta y}$$

$$\Psi_{E_x,z}\Big|^{n+1/2}_{i+1/2,j,k} = b_z \Psi_{E_x,z}\Big|^{n-1/2}_{i+1/2,j,k} + C_z \frac{H_y\Big|^{n+1/2}_{i+1/2,j,k+1/2} - H_y\Big|^{n+1/2}_{i+1/2,j,k-1/2}}{\Delta z}$$

- where

$$b_w = \exp\left(-\left(\frac{\alpha_i}{\varepsilon} + \frac{\sigma_i}{k_i \varepsilon}\right)\Delta t\right); C_w = \frac{\sigma_w}{\sigma_w k_w + \left(k_w\right)^2 \alpha_w}\left(b_w - 1\right)$$

FDTD: Advances

- Unconditional stable FDTD
  - Time step and space step can be chosen independently
  - For example, Crank Nicolson (CN) FDTD
  - Time step size can be chosen beyond CFL limit
- CN FDTD
  - Time and space derivatives are discretized by centered differences
  - Fields affected by the curl operators are averaged in time

FDTD: Advances

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

- Can convert these Maxwell's equations into matrix equation as

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{AU} + \mathbf{BU} = \left(\mathbf{A} + \mathbf{B}\right)\mathbf{U}$$

FDTD: Advances

- where

$$\mathbf{U} = \begin{bmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\varepsilon}\frac{\partial}{\partial x} & 0 \\ 0 & \frac{1}{\mu}\frac{\partial}{\partial z} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ \frac{1}{\mu}\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FDTD: Advances

- and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial x} \\ 0 & 0 & 0 & -\frac{1}{\varepsilon}\frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & -\frac{1}{\mu}\frac{\partial}{\partial y} & 0 & 0 & 0 \\ -\frac{1}{\mu}\frac{\partial}{\partial z} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\mu}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix}$$

FDTD: Advances

- Time and space derivatives are discretized by centered differences
- Fields affected by the curl operators are averaged in time

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{AU} + \mathbf{BU} = (\mathbf{A} + \mathbf{B})\mathbf{U}$$

$$\frac{\mathbf{U}^{n+1} - \mathbf{U}^{n}}{\Delta t} = (\mathbf{A} + \mathbf{B})\frac{\mathbf{U}^{n+1} + \mathbf{U}^{n}}{2}$$

FDTD: Advances

- Rearranging,

$$\therefore \mathbf{U}^{n+1} - \mathbf{U}^{n} = \left(\frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B} \right)\left(\mathbf{U}^{n+1} + \mathbf{U}^{n} \right)$$

- Combining terms,

$$\Rightarrow \mathbf{U}^{n+1} - \left(\frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B} \right)\mathbf{U}^{n+1} = \mathbf{U}^{n} + \left(\frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B} \right)\mathbf{U}^{n}$$

$$\Rightarrow \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} - \frac{\Delta t}{2}\mathbf{B} \right)\mathbf{U}^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B} \right)\mathbf{U}^{n}$$

$$\because \left(1 \pm a_1\right)\left(1 \pm a_2\right) = 1 \pm a_1 \pm a_2 \pm a_1 a_2$$

FDTD: Advances

So adding $\frac{\Delta t^2 \mathbf{AB}}{4}$ on both sides of the above equation, we have,

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A} - \frac{\Delta t}{2}\mathbf{B} + \frac{\Delta t^2 \mathbf{AB}}{4}\right)\mathbf{U}^{n+1}$$

$$= \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A} + \frac{\Delta t}{2}\mathbf{B} + \frac{\Delta t^2 \mathbf{AB}}{4}\right)\mathbf{U}^{n} + \frac{\Delta t^2 \mathbf{AB}}{4}\left(\mathbf{U}^{n+1} - \mathbf{U}^{n}\right)$$

$$\Rightarrow \left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n+1}$$

$$= \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n} + \frac{\Delta t^2 \mathbf{AB}}{4}\left(\mathbf{U}^{n+1} - \mathbf{U}^{n}\right)$$

FDTD: Advances

- Approximate implementation
- Alternating direction implicit (ADI-FDTD) method
- It drops the last term in the equation

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right)\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n+1}$$

$$=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right)\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n}+\frac{\Delta t^{2}\mathbf{AB}}{4}\left(\mathbf{U}^{n+1}-\mathbf{U}^{n}\right)$$

FDTD: Advances

- And split it into two steps
- Step 1:

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{A}\right)\mathbf{U}^{n+\frac{1}{2}}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n}$$

- Step 2:

$$\left(\mathbf{I}-\frac{\Delta t}{2}\mathbf{B}\right)\mathbf{U}^{n+1}=\left(\mathbf{I}+\frac{\Delta t}{2}\mathbf{A}\right)\mathbf{U}^{n+\frac{1}{2}}$$

FDTD: Advances

- In ADI-FDTD, computational overhead is smaller
- Because of omission of the last term,
  - the ADI-FDTD method leads to truncation error
  - which is a function of $\frac{\Delta t^2}{4}$ times the space derivatives of the fields
- Hence imposing a limit on Δt
- ADI-FDTD improves the computational efficiency at the cost of accuracy

FDTD: Advances

- Alternating direction implicit (ADI) FDTD

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \qquad \frac{\partial H_x}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \qquad \frac{\partial H_y}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \qquad \frac{\partial H_z}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

- Two-step scheme in time
  - First half time step (First procedure)
  - Second half time step (Second procedure)

FDTD: Advances

- 3 equations for 2-D TM^z are as follows

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x}\right)$$

FDTD: Advances

- First procedure:

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$E_z\Big|_{i,j}^{n+1/2} = E_z\Big|_{i,j}^{n}$$

$$+\frac{\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n+1/2} - H_y\Big|_{i-1/2,j}^{n+1/2}\right) - \frac{\Delta t}{2\varepsilon\Delta y}\left(H_x\Big|_{i,j+1/2}^{n} - H_x\Big|_{i,j-1/2}^{n}\right)$$

- Implicit:
  - How do you calculate?

FDTD: Advances

- We can use 3rd equation

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x}\right)$$

$$H_y\Big|_{i+1/2,j}^{n+1/2} = H_y\Big|_{i+1/2,j}^{n} - \frac{\Delta t}{2\mu\Delta x}\left(E_z\Big|_{i+1,j}^{n+1/2} - E_z\Big|_{i,j}^{n+1/2}\right)$$

FDTD: Advances

- Substituting equation (3) in equation (1), we have,

$$E_z\Big|_{i,j}^{n+1/2} = E_z\Big|_{i,j}^{n}$$

$$+\frac{\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n} - \frac{\Delta t}{2\mu\Delta x}\left(E_z\Big|_{i+1,j}^{n+1/2} - E_z\Big|_{i,j}^{n+1/2}\right) - \left(H_y\Big|_{i-1/2,j}^{n} - \frac{\Delta t}{2\mu\Delta x}\left(E_z\Big|_{i,j}^{n+1/2} - E_z\Big|_{i-1,j}^{n+1/2}\right)\right)\right)$$

$$-\frac{\Delta t}{2\varepsilon\Delta y}\left(H_x\Big|_{i,j+1/2}^{n} - H_x\Big|_{i,j-1/2}^{n}\right)$$

FDTD: Advances

- Rearranging

$$\alpha_1 E_z\Big|_{i-1,j}^{n+1/2} + \beta_1 E_z\Big|_{i,j}^{n+1/2} + \gamma_1 E_z\Big|_{i+1,j}^{n+1/2}$$

$$= E_z\Big|_{i,j}^{n} + \frac{\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n} - H_y\Big|_{i-1/2,j}^{n}\right) - \frac{\Delta t}{2\varepsilon\Delta y}\left(H_x\Big|_{i,j+1/2}^{n} - H_x\Big|_{i,j-1/2}^{n}\right)$$

- where

$$\alpha_1 = -\frac{\Delta t}{2\mu\Delta x}\frac{\Delta t}{2\varepsilon\Delta x}; \beta_1 = 1 - \alpha_1 - \gamma_1; \gamma_1 = -\frac{\Delta t}{2\mu\Delta x}\frac{\Delta t}{2\varepsilon\Delta x};$$

FDTD: Advances

- In matrix form

$$\begin{bmatrix} \beta_1(1) & \gamma_1(1) & 0 & \cdots & 0 \\ \alpha_1(2) & \beta_1(2) & \gamma_1(2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \alpha_1(m-1) & \beta_1(m-1) & \gamma_1(m-1) \\ 0 & \cdots & 0 & \alpha_1(m) & \beta_1(m) \end{bmatrix} \begin{bmatrix} E_z^{n+1/2}(j,1) \\ E_z^{n+1/2}(j,2) \\ \vdots \\ E_z^{n+1/2}(j,m-1) \\ E_z^{n+1/2}(j,m) \end{bmatrix} = \begin{bmatrix} r_1^n(1) \\ r_1^n(2) \\ \vdots \\ r_1^n(m-1) \\ r_1^n(m) \end{bmatrix}$$

FDTD: Advances

- Second procedure:

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$E_z\Big|_{i,j}^{n+1} = E_z\Big|_{i,j}^{n+1/2}$$

$$+\frac{\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n+1/2} - H_y\Big|_{i-1/2,j}^{n+1/2}\right) - \frac{\Delta t}{2\varepsilon\Delta y}\left(H_x\Big|_{i,j+1/2}^{n+1} - H_x\Big|_{i,j-1/2}^{n+1}\right)$$

- Implicit:
  - How do you calculate?

FDTD: Advances

- Use the 2nd equation

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y}\right)$$

$$H_x\Big|_{i,j+1/2}^{n+1} = H_x\Big|_{i,j+1/2}^{n+1/2} - \frac{\Delta t}{2\mu\Delta y}\left(E_z\Big|_{i,j+1}^{n+1} - E_z\Big|_{i,j}^{n+1}\right)$$

FDTD: Advances

- Substituting equation (2) in equation (1), we have,

$$E_z\Big|_{i,j}^{n+1} = E_z\Big|_{i,j}^{n+1/2} + \frac{\Delta t}{2\varepsilon\Delta x}\left(H_y\Big|_{i+1/2,j}^{n+1/2} - H_y\Big|_{i-1/2,j}^{n+1/2}\right)$$

$$-\frac{\Delta t}{2\varepsilon\Delta y}\left(H_x\Big|_{i,j+1/2}^{n+1/2} - \frac{\Delta t}{2\mu\Delta y}\left(E_z\Big|_{i,j+1}^{n+1} - E_z\Big|_{i,j}^{n+1}\right) - H_x\Big|_{i,j-1/2}^{n+1/2} - \frac{\Delta t}{2\mu\Delta y}\left(E_z\Big|_{i,j}^{n+1} - E_z\Big|_{i,j-1}^{n+1}\right)\right)$$

FDTD: Advances

- Rearranging

$$\alpha_2 E_z\Big|_{i,j-1}^{n+1} + \beta_2 E_z\Big|_{i,j}^{n+1} + \gamma_2 E_z\Big|_{i,j+1}^{n+1}$$

$$= \boxed{E_z\Big|_{i,j}^{n+1/2} + \frac{\Delta t}{2\varepsilon\Delta y}\left(H_y\Big|_{i+1/2,j}^{n+1/2} - H_y\Big|_{i-1/2,j}^{n+1/2}\right) - \frac{\Delta t}{2\varepsilon\Delta x}\left(H_x\Big|_{i,j+1/2}^{n+1/2} - H_x\Big|_{i,j-1/2}^{n+1/2}\right)}$$

- where

$$\alpha_2 = -\frac{\Delta t}{2\mu\Delta x}\frac{\Delta t}{2\varepsilon\Delta x}; \beta_2 = 1 - \alpha_2 - \gamma_2; \gamma_2 = -\frac{\Delta t}{2\mu\Delta x}\frac{\Delta t}{2\varepsilon\Delta x};$$

FDTD: Advances

- In matrix form

$$\begin{bmatrix} \beta_1(1) & \gamma_1(1) & 0 & \cdots & 0 \\ \alpha_1(2) & \beta_1(2) & \gamma_1(2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \alpha_1(m-1) & \beta_1(m-1) & \gamma_1(m-1) \\ 0 & \cdots & 0 & \alpha_1(m) & \beta_1(m) \end{bmatrix} \begin{bmatrix} E_z^{n+1}(j,1) \\ E_z^{n+1}(j,2) \\ \vdots \\ E_z^{n+1}(j,m-1) \\ E_z^{n+1}(j,m) \end{bmatrix} = \begin{bmatrix} r_1^{n+1/2}(1) \\ r_1^{n+1/2}(2) \\ \vdots \\ r_1^{n+1/2}(m-1) \\ r_1^{n+1/2}(m) \end{bmatrix}$$

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- How do you find the inverse of a tridiagonal matrix?
- Consider a tridiagonal matrix such as

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

- First make it Upper triangular and do back substitution

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- Conversion from Tridiagonal matrix to Upper triangular matrix
- Multiply first equation by A_{21}/A_{11} and subtract it from second equation

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} - \frac{A_{21}A_{12}}{A_{11}} & A_{23} & 0 \\ 0 & A_{32} & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - \frac{A_{21}b_1}{A_{11}} \\ b_3 \\ b_4 \end{pmatrix}$$

- Continue the process, we will get an upper triangular matrix

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- Upper triangular matrix looks like this

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

- Consider the last equation

$$x_4 = \frac{b_4}{A_{44}}; A_{33}x_3 + A_{34}x_4 = b_3 \Rightarrow x_3 = \frac{b_3 - A_{34}x_4}{A_{33}}; x_2 = \frac{b_2 - A_{23}x_3}{A_{22}}, \cdots$$