A

Newton's Law:

$$F = m\,a_{xyz} \qquad (1)$$

Where the acceleration is measured relative to an inertial reference.

There are cases: - relative to noninertial reference in an airplane or rocket, where machine elements must move in a certain way relative to the Vehicle in order to function properly. Therefore, the motion of the machine element relative to the Vehicle is known.

if, however, the vehicle is undergoing a severe maneuver relative to inertial space, we can't use (1)

⋮

Multireference analysis. Attaching xyz to the Vehicle and XYZ to inertial space

Then.

$$F = m\left[a_{xyz} + \ddot{R} + 2\omega\times V_{xyz} + \dot{\omega}\times\rho + \omega\times(\omega\times\rho)\right]$$

$$\boxed{F - m\left(\ddot{R} + 2\omega\times V_{xyz} + \dot{\omega}\times\rho + \omega\times(\omega\times\rho)\right) = m\,a_{xyz}}$$

Newton's Law in non inertial reference frame.

These terms $-m\ddot{R} - m(2\omega \times V_{xyz})$, $m(\dot{\omega} \times \rho)$, $(\omega \times (\omega \times \rho))$ ②

are → forces → inertial forces.

$-2m\omega \times V_{xyz}$ → coriolis forces

Inertial forces result in baffling actions that are sometimes contrary to our intuition.

Highly non-inertial → cases.

Fighter pilots and stunt pilots carry out actions in a cockpit of a plane while plane is undergoing severe maneuvers. Unexpected results frequently occur for flyer if they use the cockpit interior as reference for their actions.

Example 15.16

The plan view of a rotating platform is shown in Fig. 15.40. A man is seated at the position labeled A and is facing point O of the platform. He is carrying a mass of $\frac{1}{50}$ slug at the rate of 10 ft/sec in a direction straight ahead of him (i.e., toward the center of the platform). If this platform has an angular speed of 10 rad/sec and an angular acceleration of 5 rad/sec^2 relative to the ground at this instant, what force F must he exert to cause the mass to accelerate 5 ft/sec^2 toward the center?

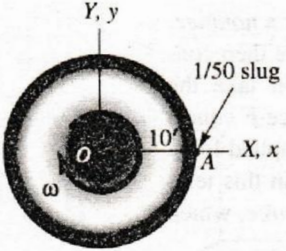


Figure 15.40. Rotating platform.

For purposes of determining inertial forces, we proceed as follows:

Fix xyz to platform.
Fix XYZ to ground.

Example 15.16 (Continued)

A. Motion of mass relative to xyz reference

$$\boldsymbol{\rho} = 10\boldsymbol{i} \text{ ft}, \quad \boldsymbol{V}_{xyz} = -10\boldsymbol{i} \text{ ft/sec}, \quad \boldsymbol{a}_{xyz} = -5\boldsymbol{i} \text{ ft/sec}^2$$

B. Motion of xyz relative to XYZ

$$\dot{\boldsymbol{R}} = 0, \quad \ddot{\boldsymbol{R}} = 0, \quad \boldsymbol{\omega} = -10\boldsymbol{k} \text{ rad/sec}, \quad \dot{\boldsymbol{\omega}} = -5\boldsymbol{k} \text{ rad/sec}^2$$

Hence,

$$\boldsymbol{a}_{XYZ} = -5\boldsymbol{i} + 2(-10\boldsymbol{k}) \times (-10\boldsymbol{i}) + (-5\boldsymbol{k}) \times 10\boldsymbol{i} + (-10\boldsymbol{k}) \times (-10\boldsymbol{k} \times 10\boldsymbol{i})$$

Therefore,

$$\boldsymbol{a}_{XYZ} = -5\boldsymbol{i} + (200\boldsymbol{j} - 50\boldsymbol{j} - 1{,}000\boldsymbol{i})$$

Employing Newton's law (Eq. 15.27) for the mass, we get

$$\boldsymbol{F} - \frac{1}{50}(200\boldsymbol{j} - 50\boldsymbol{j} - 1{,}000\boldsymbol{i}) = \frac{1}{50}(-5\boldsymbol{i})$$

Solving for F, we get

$$\boldsymbol{F} = 3\boldsymbol{j} - 20.1\boldsymbol{i} \text{ lb}$$

This force $\boldsymbol{F}$ is the *total* external force on the mass. Since the man must exert this force and also withstand the pull of gravity (the weight) in the $-\boldsymbol{k}$ direction, the force exerted by the man on the mass is

$$\boldsymbol{F}_{\text{man}} = 3\boldsymbol{j} - 20.1\boldsymbol{i} + \frac{g}{50}\boldsymbol{k} \text{ lb} \qquad \text{(a)}$$

If the platform were *not* rotating at all, it could serve as an inertial reference. Then, we would have for the total external force $\boldsymbol{F}'$:

$$\boldsymbol{F}' = \frac{1}{50}(-5\boldsymbol{i}) = -\frac{1}{10}\boldsymbol{i} \text{ lb}$$

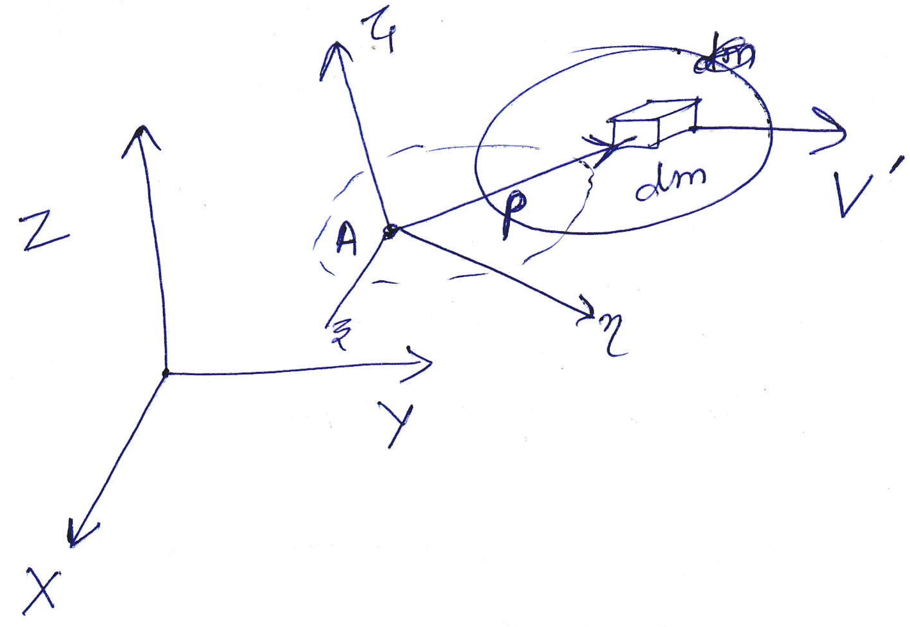
The force exerted by the man, $\mathbf{F}'_{\text{man}}$, is then

$$\boldsymbol{F}'_{\text{man}} = -\frac{1}{10}\boldsymbol{i} + \frac{g}{50}\boldsymbol{k} \text{ lb} \qquad \text{(b)}$$

This force is considerably different from that given in Eq. (a).

As a matter of interest, we note that aviators of World War 1 were required to carry out such maneuvers on a rapidly rotating and accelerating platform so as to introduce them safely to these "peculiar" effects.

Dynamics of General Rigid body Motion



Consider a rigid body moving arbitrarily relative to an inertial reference XYZ

Reference $\xi\eta\zeta$ translates with A.

A - point A in this body

dm = mass element of body at position ρ from A.

V' = Velocity of dm relative to A.
(Velocity of dm relative to reference $\xi\eta\zeta$ which translates with A relative to XYZ.

Linear momentum of dm relative to A is the linear momentum of dm relative to $\xi\eta\zeta$ translating with A.

moment of this momentum (angular momentum)

$$dH_A = \rho \times V' dm \Rightarrow \rho \times \left(\frac{d\rho}{dt}\right)_{\xi\eta\zeta} dm$$

Now A is fixed in the body (or in the hypothetical massless extension of the body) then ρ must be fixed in the body, and, accordingly

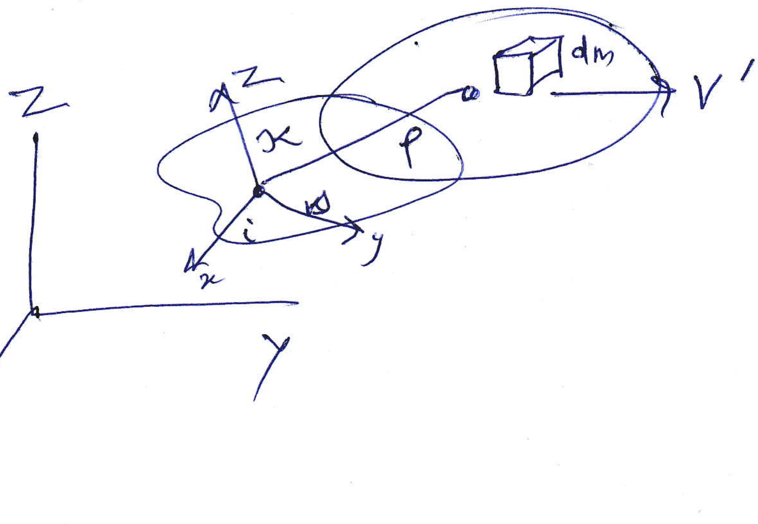
$$\left(\frac{d\rho}{dt}\right)_{\xi\eta\zeta} = \omega \times \rho$$

ω = angular velocity of the body relative to $\xi\eta\zeta$.

However, since $\xi\eta\zeta$ translates relative to XYZ, ω is the angular velocity of the body relative to XYZ as well.

$$dH_A = \rho \times (\omega \times \rho)\, dm \qquad \text{———} \quad (P)$$

Now evaluate the components relative to xyz



$$(dH_A)_x \hat{i} + (dH_A)_y \hat{j} + (dH_A)_z \hat{k}$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times [(\omega) \times (x\hat{i} + y\hat{j} + z\hat{k})] \, dm$$

ω = angular velocity vector ~~w.r.t~~ of body w.r.t. XYZ.

Ω = angular velocity vector of reference xyz w.r.t. XYZ.

$$\cancel{(dH_A)_x} = (x\hat{i} + y\hat{j} + z\hat{k}) \times \left[\begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} \right] dm$$

$$= (x\hat{i} + y\hat{j} + z\hat{k}) \times \left[i(z\omega_y - y\omega_z) - j(\omega_x z - x\omega_z) + k(y\omega_x - x\omega_y) \right] dm$$

$$= \begin{vmatrix} i & j & k \\ x & y & z \\ z\omega_y - y\omega_z & x\omega_z - \omega_x z & y\omega_x - x\omega_y \end{vmatrix}$$

$$= i\left[y^2\omega_x - yx\omega_y - zx\omega_z + z^2\omega_x \right]$$

$$- j\left[xy\omega_x - x^2\omega_y - z^2\omega_y + zy\omega_z \right]$$

$$k\left[x^2\omega_z - xz\omega_x - yz\omega_y + y^2\omega_z \right]$$

Now

$$(dH_A)_x\hat{i} + (dH_A)_y\hat{j} + (dH_A)_z\hat{k} =$$

$$\Big[\, i\left[(y^2+z^2)\omega_x - xy\,\omega_y - zx\,\omega_z\right]$$
$$+ j\left[-xy\,\omega_x + (x^2+z^2)\omega_y - zy\,\omega_z\right]$$
$$+ k\left[-xz\,\omega_x - yz\,\omega_y + (x^2+y^2)\omega_z\right]\Big]dm$$

Now Integrate all over the domain

$$\cancel{(H_A)_x\hat{i} + (H_A)_y\hat{j} + (H_A)_z\hat{k}}$$

$$\cancel{= \left[\omega_x \iiint_V (x^2+y^2)\,dm\,dV - \omega_y \int_V xy\,dm\,dV - \omega_z \int_V zx\,dm\,dV\right]}\hat{i}$$

$$\cancel{+\left[-\omega_x \int_V xy\,dm\,dV + \omega_y \int_V (x^2+z^2)\,dm\,dV\right.}$$

$$(H_A)_x\hat{i} + (H_A)_y\hat{j} + (H_A)_z\hat{k}$$

$$= \begin{bmatrix} (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\hat{i} \\ +(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\hat{j} \\ +(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\hat{k} \end{bmatrix}$$

$$\iiint_{dm} (y^2 + z^2)\, dm = I_{xx}$$

Now

$$(H_A)_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$(H_A)_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$

$$(H_A)_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

~~Euler's Equation of Motion~~

~~Now~~ We can arrange in terms of inertia matrix

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & I_{yz} \\ -I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

It can be shown that there is one unique orientation of axes x-y-z for a given origin for which the product of inertia vanish and moment of inertia I_{xx}, I_{yy}, I_{zz} take on stationary values

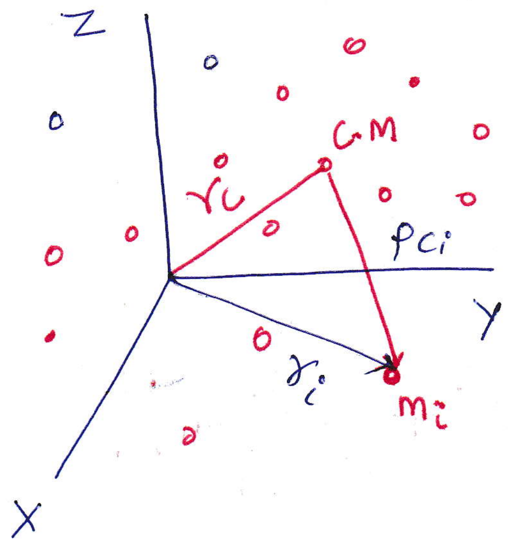
$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Kinetic Energy of a Rigid Body

Kinetic Energy of a particle

$$K.E = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{r}^2$$

Kinetic Energy of a system of particles:



$$r_i = r_c + \rho_{ci} \quad \text{——— (1)}$$

Differentiating w.r.t. time.

$$\frac{dr_i}{dt} = \frac{dr_c}{dt} + \frac{d\rho_{ci}}{dt}$$

$$V_i = V_c + \dot{\rho}_{ci}$$

Where $\dot{\rho}_{ci} \Rightarrow$ motion of the ith particle relative to mass center.

$$K.E = \sum_{i=1}^{n} \frac{1}{2} m_i (V_c + \dot{\rho}_{ci})^2 = \sum_{i=1}^{n} \frac{1}{2} m_i (V_c + \dot{\rho}_{ci}) \cdot (V_c + \dot{\rho}_{ci})$$

$$K.E = \frac{1}{2}\sum_{i=1}^{n} m_i V_c^2 + \sum_{i=1}^{n} m_i V_c \cdot \dot{\rho}_{c_i} + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\rho}_{c_i}^2$$

Now

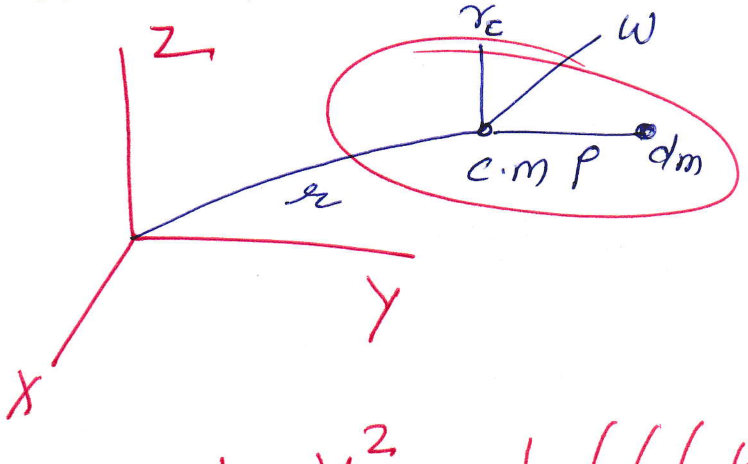
$$K.E = \frac{1}{2}\sum_{i=1}^{n} m_i V_c^2 + V_c \cdot \sum_{i=1}^{n} m_i \dot{\rho}_{c_i} + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\rho}_{c_i}^2$$

$$\underline{\underline{V_c \cdot \frac{d}{dt}\sum_{i=1}^{n}(m \rho_{c_i})}}$$

$$\boxed{\sum_{i=1}^{n} m \rho_{c_i} = 0}$$ — first moment of mass of the system about the center of mass

$$\therefore \boxed{K.E = \frac{1}{2} m V_c^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\rho}_{c_i}^2}$$

Let us consider the above equation applied to rigid body. In such case, the velocity of any particle relative to mass center becomes -

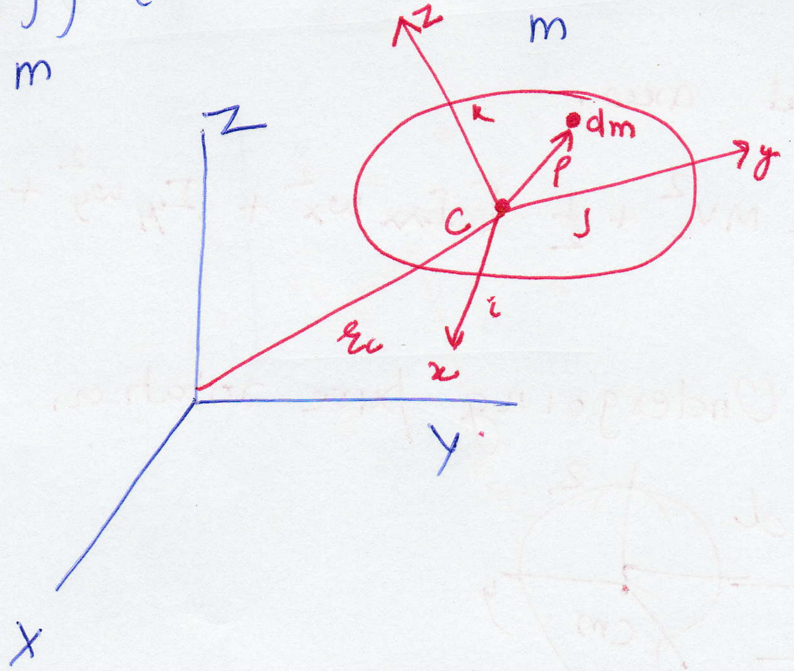


$$\underline{\underline{\dot{\rho}_i = \omega \times \rho_i}}$$

$$K.E = \frac{1}{2} m V_c^2 + \frac{1}{2}\iiint_m (\omega \times \rho_i)^2 dm$$

Now choose a set of orthogonal directions xyz at the center of mass, so we can carry out the preceding integration in terms of the scalar components ω and ρ.

$$\iiint_m (\omega \times \rho)^2 dm = \iiint_m (\omega \times \rho) \cdot (\omega \times \rho) dm$$



$$\iiint_m (\omega \times \rho)^2 dm = \iiint_m \Big[(\omega_x i + \omega_y j + \omega_z k) \times (xi + yj + zk)\Big] \cdot \Big[(\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})\Big]$$

Carrying out first cross-product and dot product in the integrand and collecting terms.

$$= \Big[\iiint (z^2 + y^2) dm\Big]\omega_x^2 - \Big[\iiint_m (xy) dm\Big]\omega_x \omega_y - \Big[\iiint_m xz\, dm\Big]\omega_x\omega_z$$

$$- \iiint_m (yx) dm\, \omega_y \omega_x + \Big(\iiint_m (x^2 + z^2) dm\Big)\omega_y^2 - \Big[\iiint_m yz\, dm\Big]\omega_y\omega_z$$

$$- \Big[\iiint_m zx\, dm\Big]\omega_z\omega_x - \Big(\iiint_m zy\, dm\Big)\omega_z\omega_y + \Big[\iiint_m (x^2 + y^2) dm\Big]\omega_z^2$$

XIV

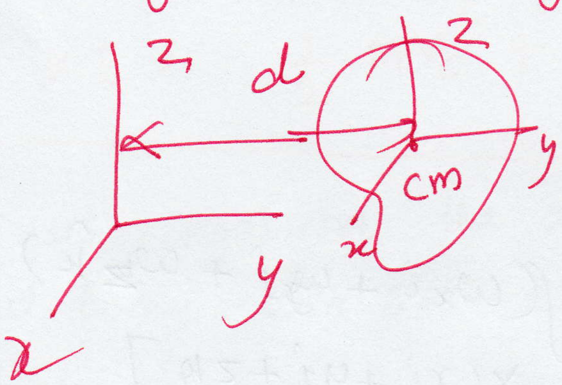
Integrals are the moments and product of Inertia for the xyz reference.

$$\iiint |\omega \times \rho|^2 \, dm = I_{xx}\omega_x^2 - I_{xy}\omega_x\omega_y - I_{xz}\omega_x\omega_z - I_{yx}\omega_x\omega_y + I_{yy}\omega_y^2 - I_{yz}\omega_y\omega_z - I_{zx}\omega_z\omega_x - I_{zy}\omega_z\omega_y + I_{zz}\omega_z^2$$

For Principal axes.

$$K.E = \frac{1}{2} M V_c^2 + \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 \right]$$

Rigid body Undergoing pure rotation



rigid body undergoing pure rotation relative to XYZ with angular velocity ω

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega$$

$$K.E = \frac{1}{2} m V_c^2 + \frac{1}{2} I_{zz}\omega^2$$