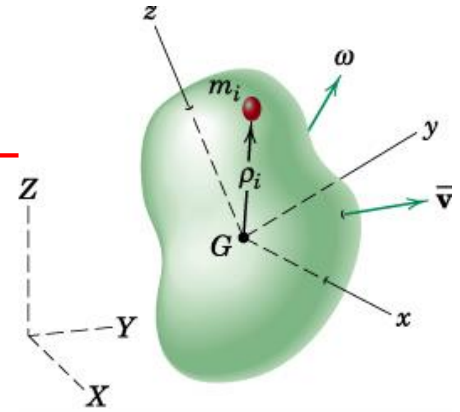
3D Kinetics of Rigid Bodies



Angular Momentum

$$\mathbf{H}_G = \int [\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})]\, dm$$

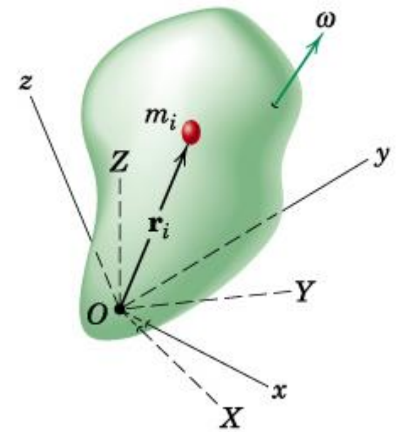
$$\mathbf{H}_O = \int [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, dm$$

Expression for **H**:

$$\begin{aligned} \mathbf{H} = & \ (\ I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\mathbf{i} \\ & + (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\mathbf{j} \\ & + (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\mathbf{k} \end{aligned}$$

Components of **H**:

$$\begin{aligned} H_x &= \ \ I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$



Inertia Matrix

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yz} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

Principal axes of inertia

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$\mathbf{H} = I_{xx}\omega_x\mathbf{i} + I_{yy}\omega_y\mathbf{j} + I_{zz}\omega_z\mathbf{k}$$

Kinetic Energy

$$T = \frac{1}{2}m\bar{v}^2 + \Sigma\frac{1}{2}m_i|\dot{\boldsymbol{\rho}}_i|^2$$

3-D Kinetics of Rigid Bodies

General Momentum Equations

For systems of particles of constant mass:

$$\Sigma \mathbf{F} = \dot{\mathbf{G}}$$
$$\Sigma \mathbf{M} = \dot{\mathbf{H}}$$

: The terms are considered either @ fixed point *O* or @ the mass center *G*
: Derivative of **H** was taken wrt an absolute coordinate system

:: For a moving coordinate system *x*-*y*-*z* with angular velocity **Ω**, the moment relation:

$$\left(\frac{d\mathbf{V}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{V}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{V}$$

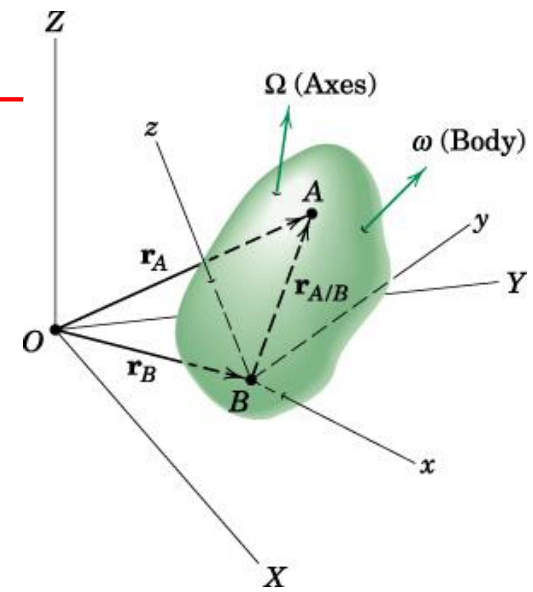
→

$$\Sigma \mathbf{M} = \left(\frac{d\mathbf{H}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{H} = (\dot{H}_x \mathbf{i} + \dot{H}_y \mathbf{j} + \dot{H}_z \mathbf{k}) + \mathbf{\Omega} \times \mathbf{H}$$

:: 1st term represents the change in magnitude of the components of **H**
:: 2nd term represents the change in direction of the components of **H**

$$\begin{aligned} \Sigma \mathbf{M} = &(\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\mathbf{i} \\ &+ (\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\mathbf{j} \\ &+ (\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\mathbf{k} \end{aligned}$$

3-D Kinetics of Rigid Bodies



General Momentum Equations

$$\Sigma \mathbf{M} = (\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\mathbf{i} + (\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\mathbf{j} + (\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\mathbf{k}$$

$$\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

For reference axes attached to the body,

- Moments and products of inertia will become invariant with time
- $\Omega = \omega$

→ Three scalar components of the general moment equation:

$$\begin{aligned} \Sigma M_x &= \dot{H}_x - H_y\omega_z + H_z\omega_y \\ \Sigma M_y &= \dot{H}_y - H_z\omega_x + H_x\omega_z \\ \Sigma M_z &= \dot{H}_z - H_x\omega_y + H_y\omega_x \end{aligned}$$

3-D Kinetics of Rigid Bodies

General Momentum Equations

General Form

$$\Sigma\mathbf{M} = (\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\mathbf{i} + (\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\mathbf{j} + (\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\mathbf{k}$$

Axes attached to the body ($\omega=\Omega$)

$$\Sigma M_x = \dot{H}_x - H_y\omega_z + H_z\omega_y$$
$$\Sigma M_y = \dot{H}_y - H_z\omega_x + H_x\omega_z$$
$$\Sigma M_z = \dot{H}_z - H_x\omega_y + H_y\omega_x$$

For the **reference axes coinciding** with the **principal axes of inertia**,

- origin either at the mass center *G* or at a point *O* fixed to the body and fixed in space
- $I_{xy} = I_{yz} = I_{xz} = 0$

Moment Eqns:

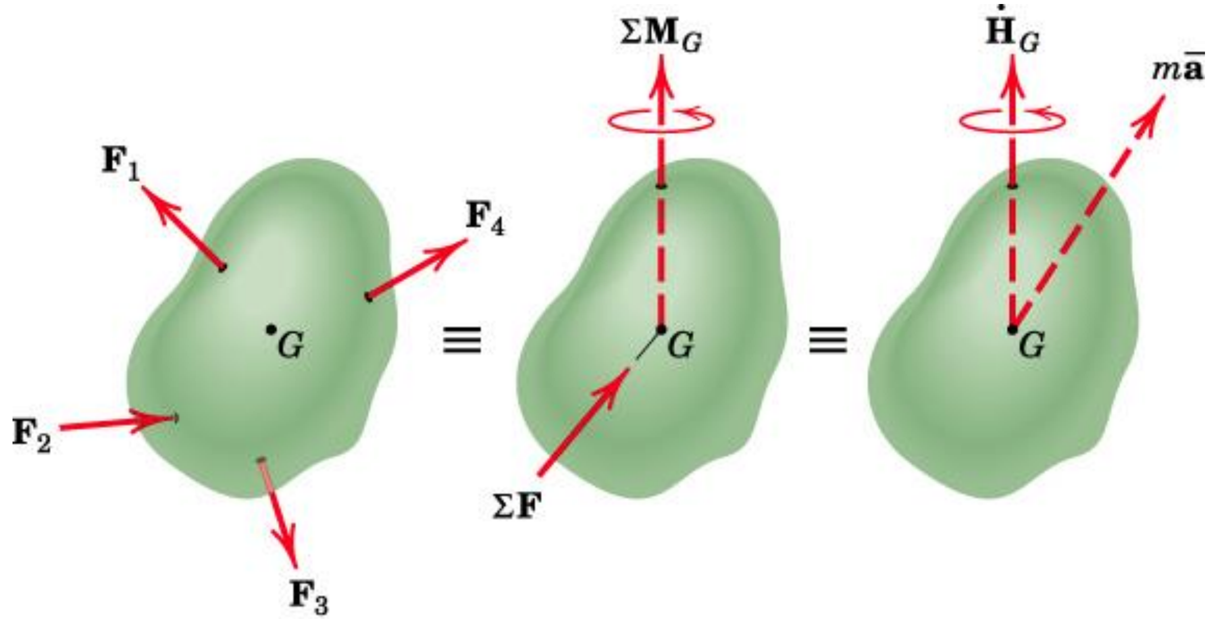
$$\Sigma M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z$$
$$\Sigma M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x$$
$$\Sigma M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y$$

Euler's Equations

•Principal Axes of Inertia ($I_{xy} = I_{yz} = I_{xz} = 0$)

3-D Kinetics of Rigid Bodies

Energy Equations



: **Rate of work** done by the **resultant force** and the **resultant couple**:

$\sum \mathbf{F} \cdot \bar{\mathbf{v}}$ and $\sum \mathbf{M}_G \cdot \boldsymbol{\omega}$

: Total work done during the time interval = Change in Kinetic Energy

$$\int_{t_1}^{t_2} \Sigma \mathbf{F} \cdot \bar{\mathbf{v}}\, dt = \frac{1}{2} \bar{\mathbf{v}} \cdot \mathbf{G} \Big|_1^2 \qquad \int_{t_1}^{t_2} \Sigma \mathbf{M}_G \cdot \boldsymbol{\omega}\, dt = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_G \Big|_1^2$$

$$T = \frac{1}{2} \bar{\mathbf{v}} \cdot \mathbf{G} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_G$$

- Total change in Translational KE and change in Rotational KE for the interval during which $\sum \mathbf{F}$ or $\sum \mathbf{M}_G$ acts

3-D Kinetics of Rigid Bodies

Energy Equations

: For rigid body motion in 3-D,

$$U'_{1\text{-}2} = \Delta T + \Delta V$$

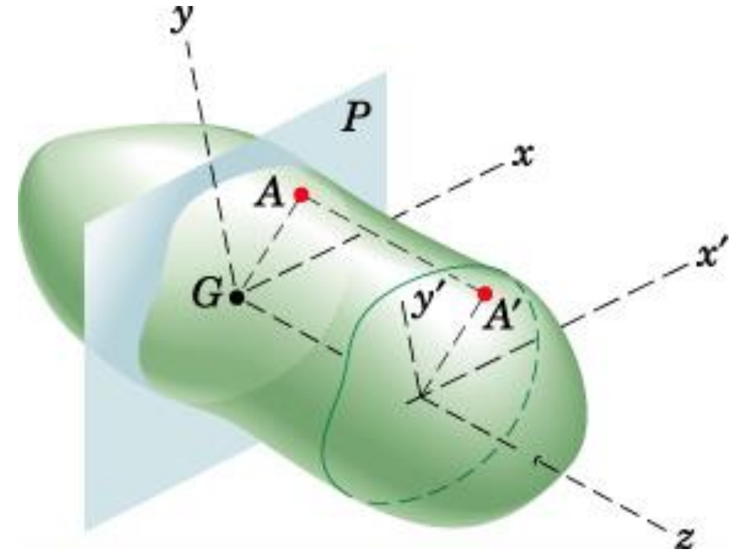
: Highly advantageous if the **initial** and **final end point** conditions of motion are analyzed

:: Two important applications

- **Parallel Plane Motion**
- **Gyroscopic Motion**

3-D Kinetics of Rigid Bodies

Parallel Plane Motion



: All points in a rigid body move in planes parallel to a fixed plane *P*

: Every line in a body normal to the fixed plane remains parallel to itself at all times

- Mass center *G* as the origin of *x-y-z* system attached to the body
- x-y plane coincides with the plane of motion *P*

:: $\omega_x = \omega_y = 0$, $\omega_z \neq 0$

:: $H_x = -I_{xz}\,\omega_z$, $H_y = -I_{yz}\,\omega_z$, $H_z = I_{zz}\,\omega_z$

→ The general moment relations:

$$\Sigma M_x = -I_{xz}\dot{\omega}_z + I_{yz}\omega_z^{\,2}$$
$$\Sigma M_y = -I_{yz}\dot{\omega}_z - I_{xz}\omega_z^{\,2}$$
$$\Sigma M_z = I_{zz}\dot{\omega}_z$$

Applicable for origin at mass center or for any origin on a fixed axis of rotation.

Three independent force eqns of motion: $\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma F_z = 0$

3-D Kinetics of Rigid Bodies

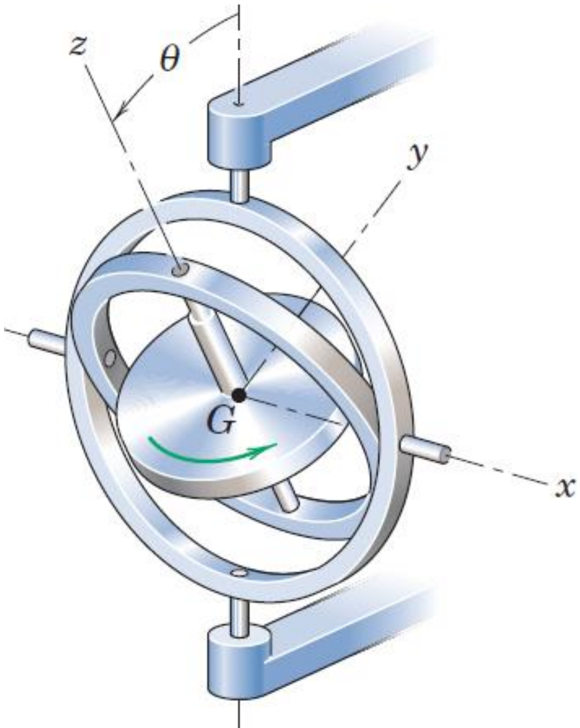
Gyroscopic Motion: Steady Precession

Gyroscopic Motion

- When the **axis**, **about which a body is spinning**, is **itself rotating about another axis**

Precession

- **Change** in the **orientation** of the **rotational axis** of a rotating body (change in direction of the rotation axis)
- Steady Precession (steady rate) included in the current course

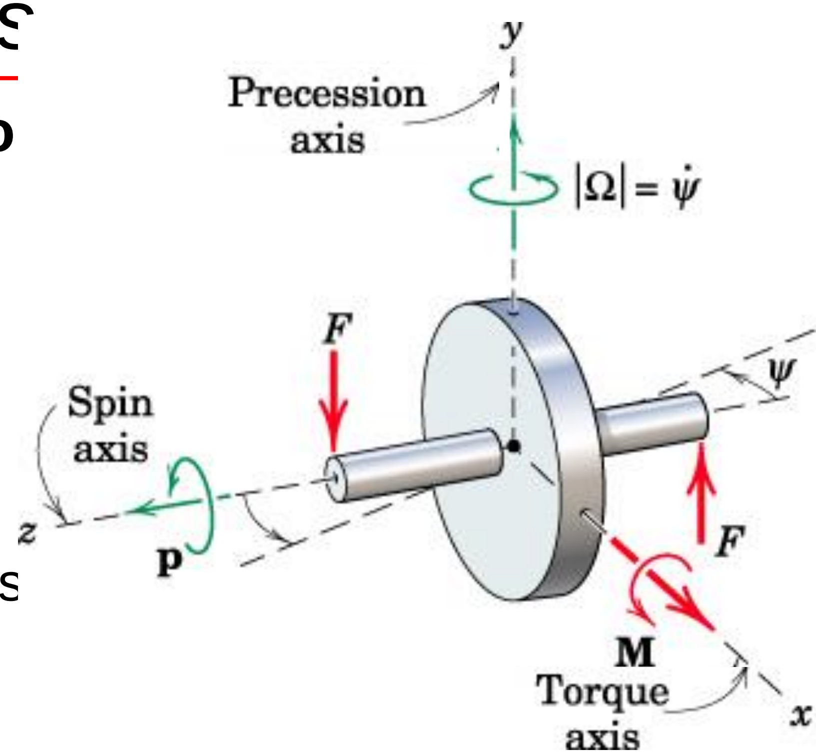


Important applications:

- Inertial guidance systems
  - Gyroscopic compass
- Directional control devises
- Stabilizing device
  - Prevention of rolling of ship in sea

3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precessio



Simplified Approach

: A symmetrical rotor spinning @ *z*-axis with **large** angular velocity *P* (spin velocity)

: Apply two forces *F* to the rotor axle to form a couple **M** with vector directed along *x*-axis

→ the rotor shaft will rotate in the *x*-*z* plane @ the *y*-axis in the sense indicated with a relatively slow angular velocity Ω

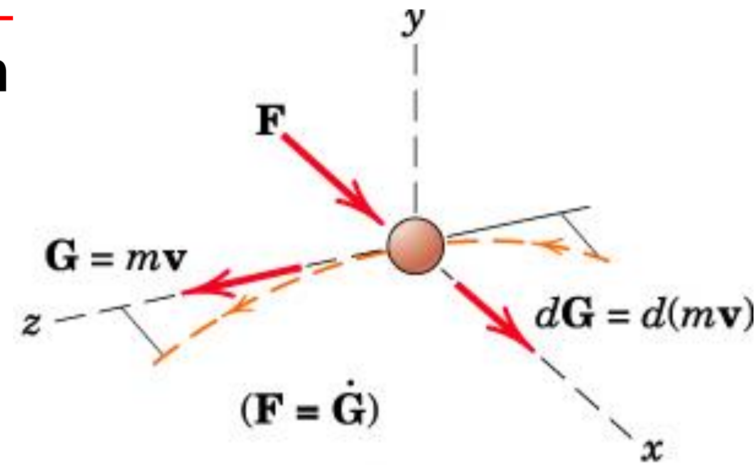
→ Precession Velocity $\Omega = \dot{\psi}$

→ No observed rotation @ *x*-axis in the sense of **M**

Three major axes: Spin Axis (**p**)
Torque Axis (**M**)
Precession Axis (**Ω**)

3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession



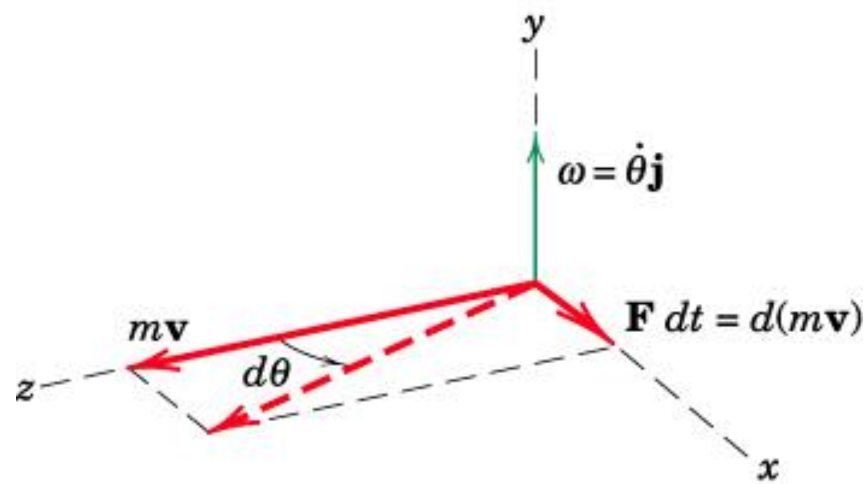
: A particle of mass *m* moving in *x-z* plane
with constant speed |**v**| = *v*

: Application of a force **F** normal to its linear
momentum **G** = *m***v**
→ change *d***G** = *d*(*m***v**) in its momentum
→ *d***v** :: same dirn as **F** as per Newton's second law **F** = **G**

: $\mathbf{F} = \dot{\mathbf{G}} \equiv \mathbf{F}dt = d\mathbf{G}$

In the limit: $\tan d\theta \sim d\theta = F\,dt/mv$

→ $F = mv\dot{\theta}$



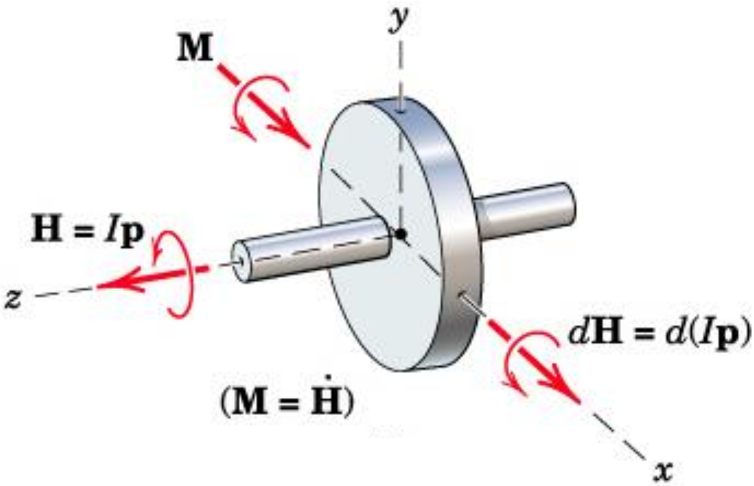
Since $\boldsymbol{\omega} = \mathbf{j}\,\dot{\theta}$ → Rotation @ *y* because of change in dirn of linear momentum

→ $\mathbf{F} = m\,\boldsymbol{\omega} \times \mathbf{v}$

→ Vector equivalent of scalar reln $F_n = ma_n$ for normal force on the particle

3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession



Now with respect to mass centre
or fixed point O

$\mathbf{M} = \dot{\mathbf{H}}$

For the symmetrical rotor,

: With high **p** and low **Ω** @ y-axis, $\mathbf{H} = I\,\mathbf{p}$
where $I = I_{zz}$ is the moment of inertia of the rotor about the spin axis

: The small component of angular momentum @ *y*-axis is neglected

: Application of couple **M** normal to **H**
→ change $d\mathbf{H} = d(I\mathbf{p})$ in angular momentum

: $d\mathbf{H}$ and thus $d\mathbf{p}$ are along the dirn of couple **M**

: Since $\mathbf{M} = \dot{\mathbf{H}}$, $\mathbf{M}dt = d\mathbf{H}$
→ **M**, **H**, $d\mathbf{H}$ are analogous to **F**, **G**, $d\mathbf{G}$

3-D Kinetics of Rigid Bodies

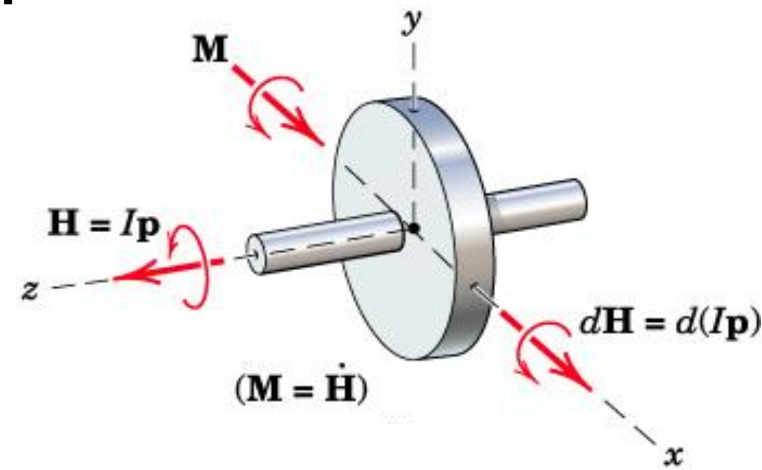
Gyroscopic Motion: Steady Precession

M causes a change in **H**

→ change of direction of **H**
towards *d***H** or **M** (torque) axis

→ rotation vector changes in the dirn of **M**

→ Precession of spin axis @ *y*-axis



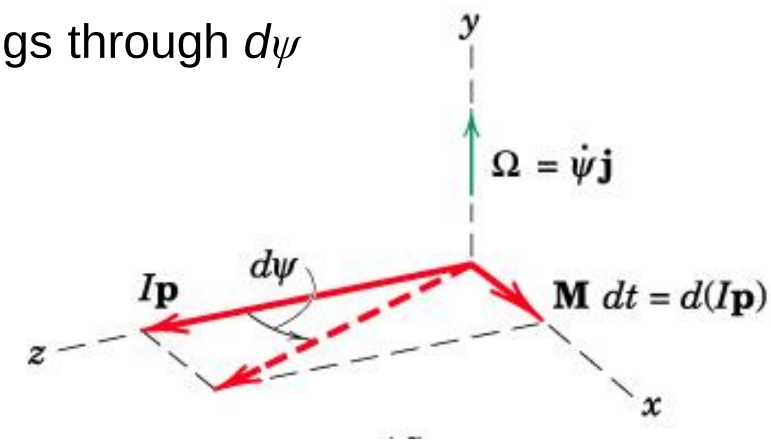
→Ang momentum (spin axis) tend to rotate toward the torque axis
→Precession

- During time *dt*, angular momentum vector $I\mathbf{p}$ swings through $d\psi$
- In the limit, tan $d\psi = d\psi$

$$d\psi = \frac{M\,dt}{Ip} \rightarrow M = I\frac{d\psi}{dt}p$$

- Since magnitude of precession velocity $\Omega = d\psi/dt$

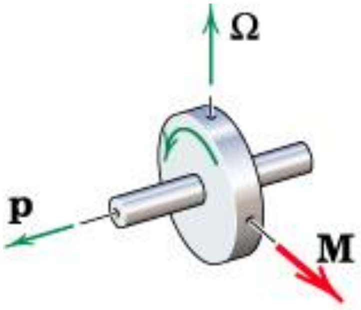
$$M = I\Omega p \rightarrow \mathbf{M} = I\boldsymbol{\Omega} \times \mathbf{p}$$



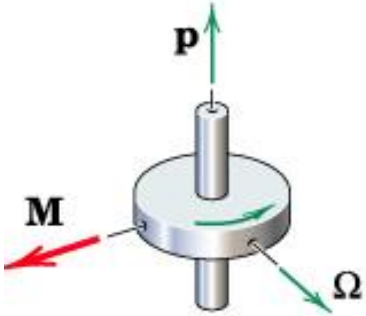
3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession

$\mathbf{M} = I\mathbf{\Omega} \times \mathbf{p}$ analogous to $\mathbf{F} = m\,\boldsymbol{\omega} \times \mathbf{v}$



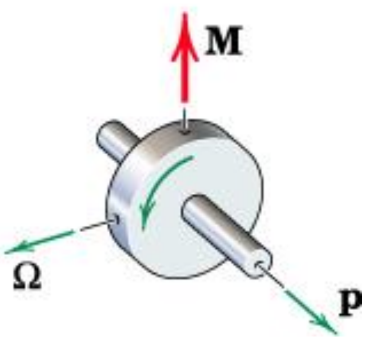
- Both eqns are applicable when moments are taken @ mass center or @ a fixed point on axis of rotation.
- *I* is the moment of inertia of rotating body taken @ spin axis



M, **Ω**, and **p** are mutually perpendicular vectors in this case

- Spin vector always tend to rotate toward the torque vector
- Orientations of the three vectors should be consistent with their correct order

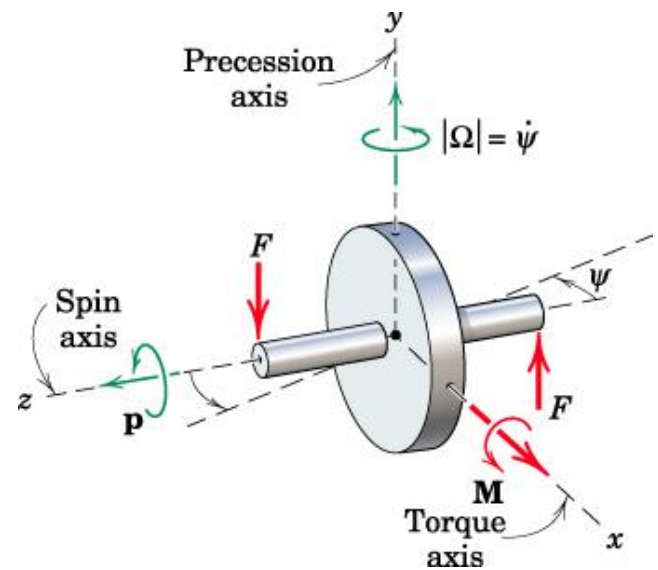
→Extremely important. Otherwise precession will be in the opposite direction.



Assumption:
The component of angular momentum neglected @ the *y*-axis which accompanies the slow precession

3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession



Incorporating ang momentum @ *y*-axis

Since the rotor has a rotational velocity @ *y*-axis (Precession Velocity), and also the rotor has a moment of inertia @ *y*-axis

→additional comp of angular momentum @ *y*-axis

→Two components: $H_z = I\,p$ and $H_y = I_0\,\Omega$

$I = I_{zz}$ @ spin axis, and $I_0 = I_{yy}$ @ precession axis

→ Total angular momentum = **H**

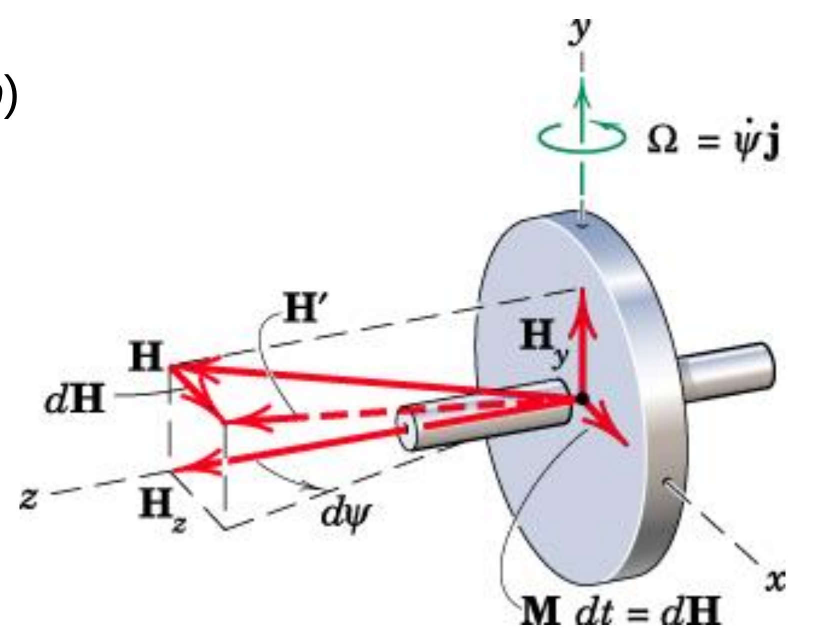
→Change in H: $d\mathbf{H} = \mathbf{M}dt$

→Precession during *dt*: $d\psi = M\,dt/H_z = M\,dt/\,(I\,p)$

→These are same eqns as obtained before.

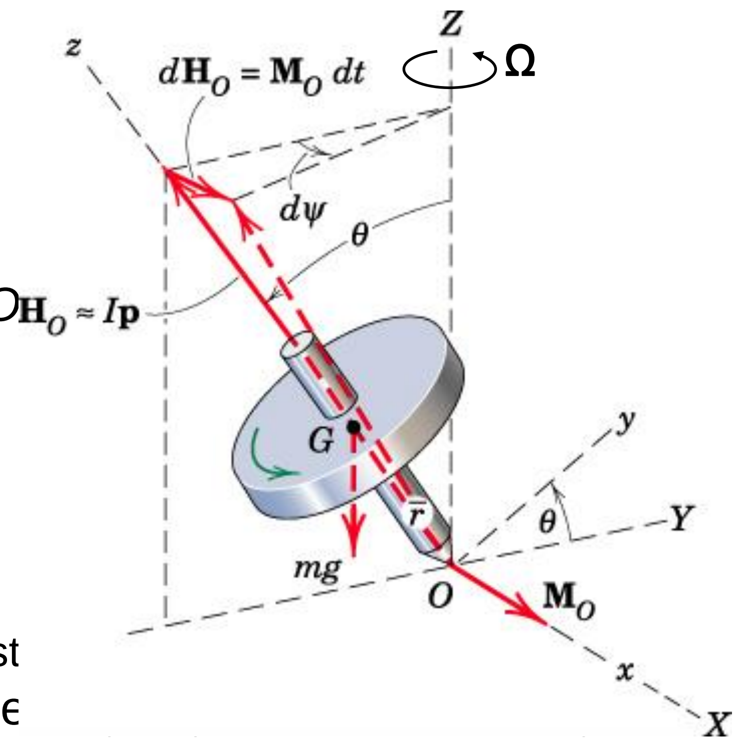
→ $$M = I\Omega p$$ is still valid (I is @ spin axis)

(this eqn gives an exact description of motion as long as the spin axis is perpendicular to the precession axis)



3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession



Influence of Ω (precession velocity) on momentum relations

- Steady precession of a symmetrical top spinning @ its axis (*z*) with high ang vel *p* & supported at *O*
- Precession @ vertical *Z*-axis
- Spin axis makes an angle *θ* with precession axis
- Neglect small ang momentum due to precession

→ Total Ang momentum is @ the axis of the top associated with spin only: $\mathbf{H}_o = I\,\mathbf{p}$

- Moment @ *O*, $M_o = mg\bar{r}\sin\theta$ due to wt of top ($\bar{r}$ is dist
- During *dt*, *θ* remains unchanged, & $\mathbf{H}_o$ has a change

→Increment in precession angle around *Z*-axis: $$d\psi = \frac{M_O\, dt}{Ip \sin\theta}$$

Substituting for M_o and $\Omega = d\psi/dt$

→$mg\bar{r}\sin\theta = I\,\Omega p \sin\theta$ → $mg\bar{r} = I\,\Omega p$ (independent of *θ*)

Introducing radius of gyration: $I = mk^2$ → **Precession Velocity**:

$$\Omega = \frac{g\bar{r}}{k^2 p}$$

(*I* and *k* are taken @ spin axis)

3-D Kinetics of Rigid Bodies

Gyroscopic Motion: Steady Precession

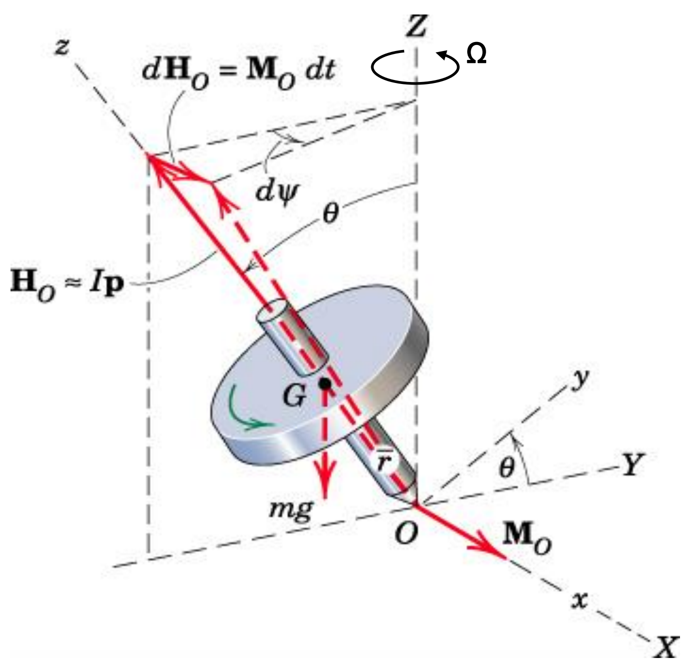
→ Based on the assumption that the angular momentum associated with Ω is negligible as compared with that associated with *p*

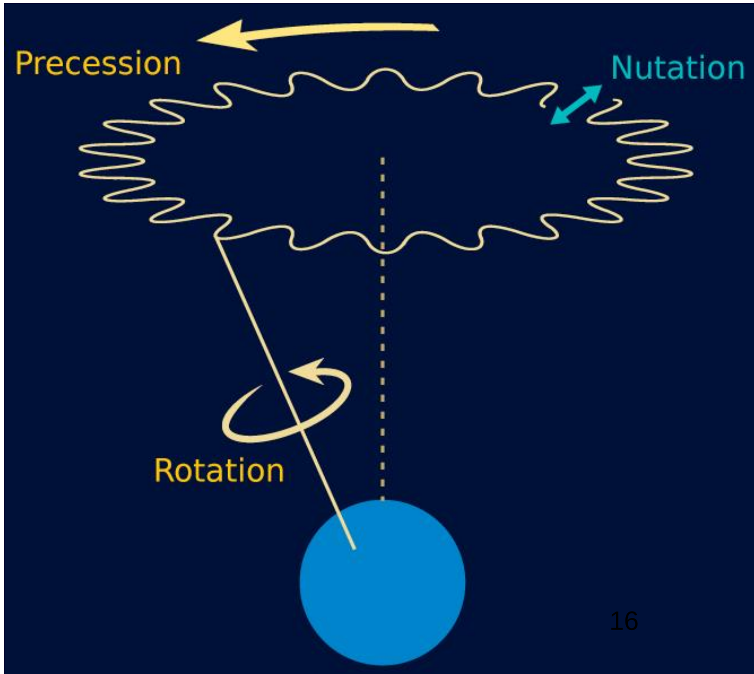
$$\Omega = \frac{g\bar{r}}{k^2 p}$$

→ Top will have steady precession with constant *θ* only when this eqn is satisfied

→ Else, unsteady precession :: oscillation of *θ* (increases with reduction in spin velocity)

→ **Nutation**

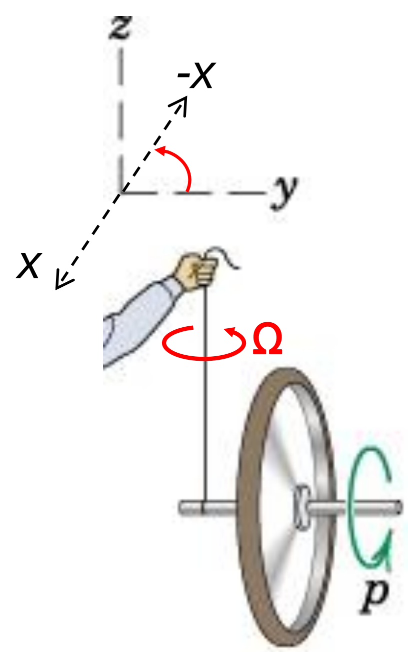




Example (1) on Gyroscopic Motion

A rapidly spinning wheel is suspended vertically with a string attached to one end of its horizontal axle. Describe the precession motion of the wheel.

Solution:



: Weight of the wheel will produce torque **M** @ -ve *x*-axis
: Angular momentum due to spinning will act @ +ve *y*-axis
→ change in ang momentum will be towards torque axis
→ Precession will take place @ +ve *z*-axis
(counter-clockwise precession when viewed from top)

Alternatively:

Weight of the wheel will produce torque **M** @ -ve *x*-axis: $\mathbf{M} = -M\mathbf{i}$

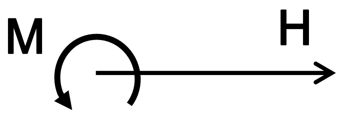
$$\mathbf{M} = I\mathbf{\Omega} \times \mathbf{p}$$

→ $-M\,\mathbf{i} = I\,\mathbf{\Omega} \times p\,\mathbf{j}$ → naturally, $\mathbf{\Omega}$ is in +$\mathbf{k}$ direction.

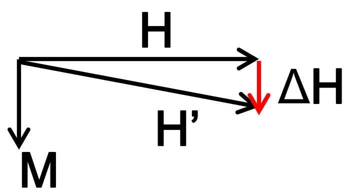
Example (2) on Gyroscopic Motion

The wheel is rapidly spinning with angular speed *p*. The axle of the wheel is held in horizontal position, and then it is attempted to tilt the spinning axis upward in the vertical plane. What motion tendency of the wheel assembly will be sensed by the person holding the axle?

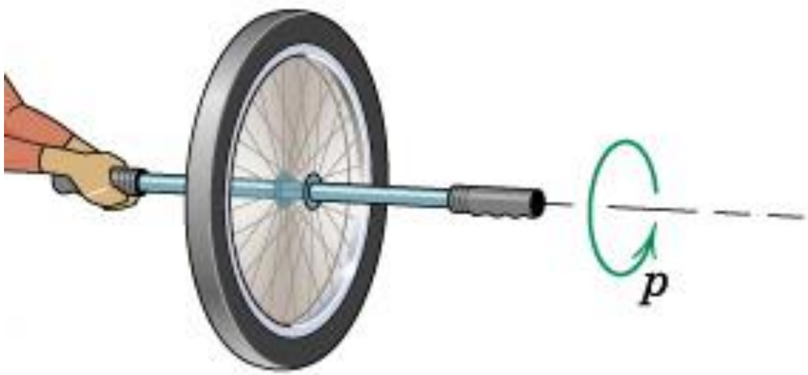
Solution:



Side View



Overhead View



M is the moment exerted on the axle by the person

H is the angular momentum of the wheel

Δ**H** is in the direction of **M**

H' is the new angular momentum

→The person will sense the tendency of the wheel to rotate to his right