Plane Kinematics of Rigid Bodies

Rigid Body

- A system of particles for which the distances between the particles remain unchanged.
- This is an ideal case. There is always some deformation in materials under the action of loads. This deformation can be neglected if the changes in the shape are small compared to the movement of the body as a whole.

Particle Kinematics

- Developed the relationships governing the disp, vel, and accln of points as they move along straight or curved paths.

Rigid Body Kinematics

- Same relationships will be used. In addition, rotational motion of rigid bodies will also be accounted for.
- Involves both linear and angular disp, vel, and accln.
- Discussion will be restricted to motion in a Single Plane (Plane Motion).
  - When all parts of the body move in parallel planes
  - Plane containing mass center is generally considered as plane of motion and the body is treated as a thin slab whose motion is confined to the plane of the slab.

Plane Kinematics of Rigid Bodies

Plane Motion

Translation

No rotation of any line in body. Motion of the body specified by motion of any point in the body ≈ Motion of a single particle.

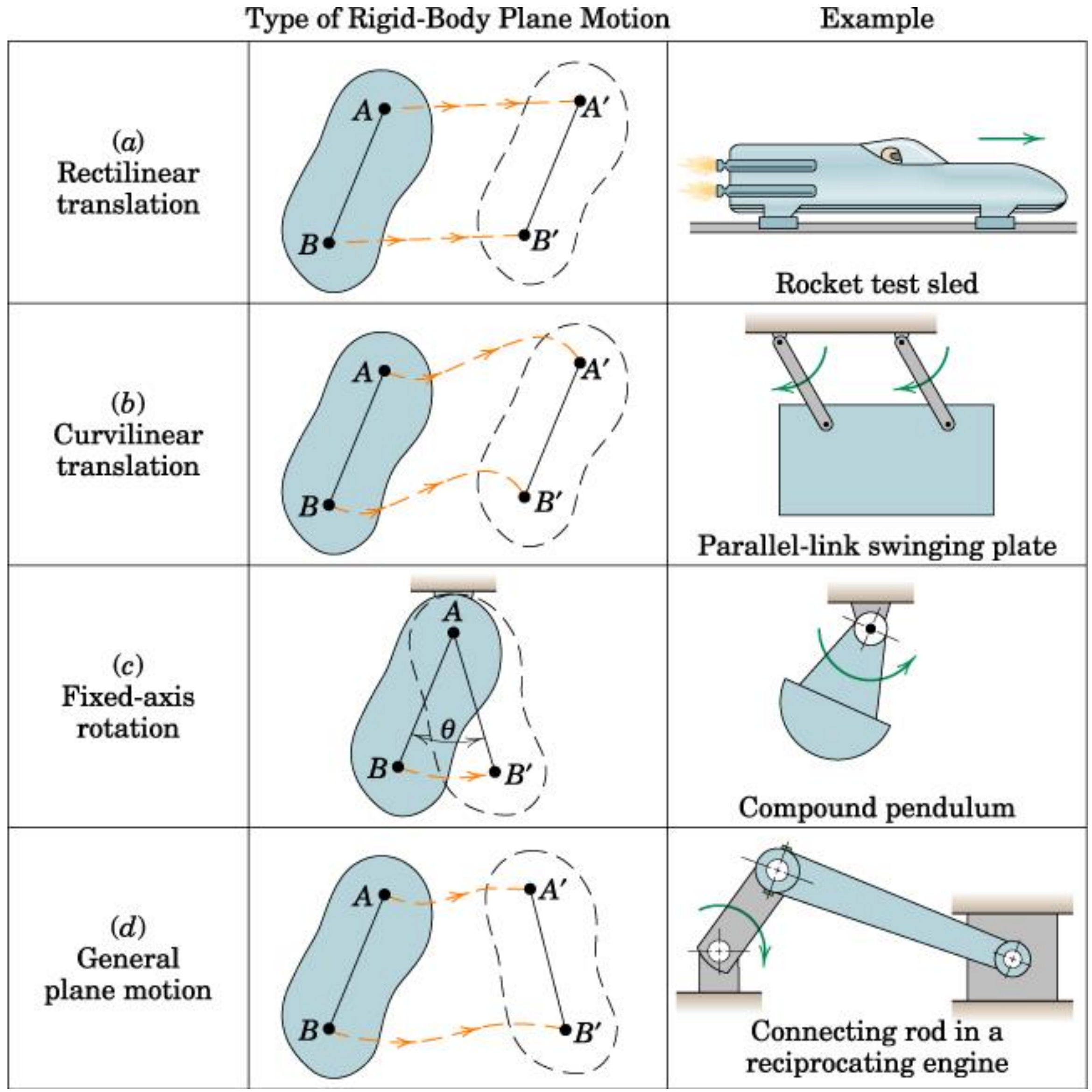
Rotation @ a Fixed Axis

All particles move in circular paths @ axis of rotn. All lines perpendicular to the axis of rotn rotate through the same angle.

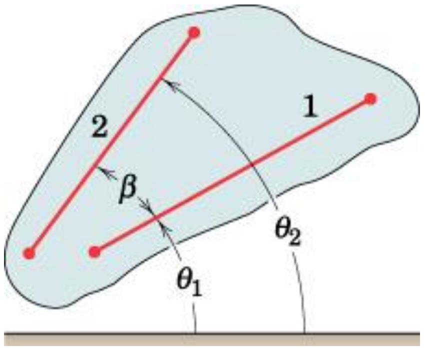
General Planar Motion

Combination of translation and rotation

The actual paths of all particles in the body are projected on to a single plane of motion.



Plane Kinematics of Rigid Bodies



Rotation

- Described by angular motion

Consider plane motion of a rotating rigid body

$\theta_2 = \theta_1 + \beta$ since β is invariant

Therefore, $\dot{\theta}_2 = \dot{\theta}_1$ and $\ddot{\theta}_2 = \ddot{\theta}_1$

And, during a finite interval: $\Delta\theta_2 = \Delta\theta_1$

→ All lines on a rigid body in its plane of motion have the same angular displacement, same angular velocity (ω), and same angular acceleration (α).

For plane Rotation:

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

$$\alpha = \frac{d\omega}{dt} = \dot{\omega} \quad \text{or} \quad \alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

$$\omega\, d\omega = \alpha\, d\theta \quad \text{or} \quad \dot{\theta}\, d\dot{\theta} = \ddot{\theta}\, d\theta$$

For rotation with constant angular acceleration, integrals of these eqns give:

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = \omega_0{}^2 + 2\alpha(\theta - \theta_0)$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

At $t = 0$, $\theta = \theta_0$, $\omega = \omega_0$

+ve dirn for ω and α is the same as that chosen for θ

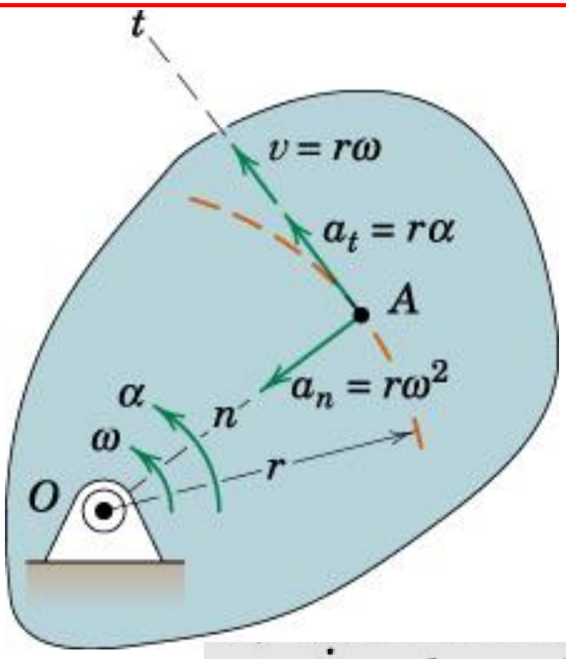
Particle Kinematics → Rigid Body Kinematics

(s, v, a) → (θ, ω, α)

Plane Kinematics of Rigid Bodies

Rotation @ a Fixed Axis

•All points (except those on the axis) move in concentric circles @ the fixed axis normal to the plane of the figure



$\omega = \dot{\theta}$ and $\alpha = \dot{\omega} = \ddot{\theta}$

Vel of *A* using *n-t* for circular motion:

$$v_n = \dot{r} = 0, \ \therefore v = v_t = r\dot{\theta} = r\omega$$

For particle circular motion

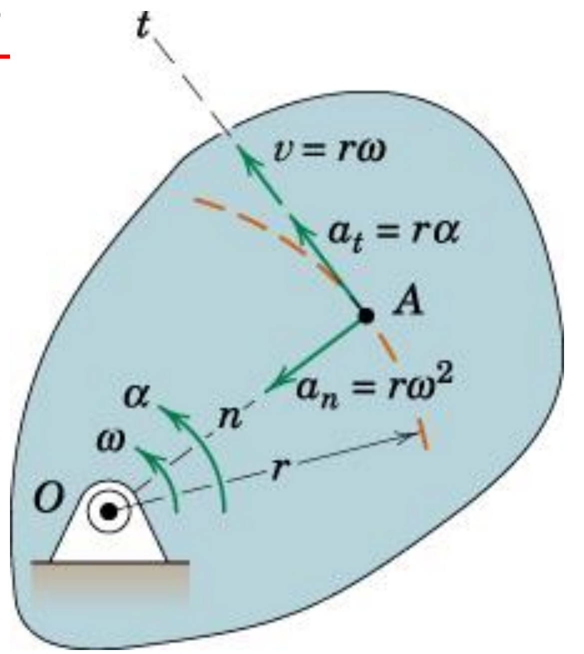
$$v = r\dot{\theta}$$
$$a_n = v^2/r = r\dot{\theta}^2 = v\dot{\theta}$$
$$a_t = \dot{v} = r\ddot{\theta}$$

→

For body rotational motion

$$v = r\omega$$
$$a_n = r\omega^2 = v^2/r = v\omega$$
$$a_t = r\alpha$$

Plane Kinematics of Rigid Bodies



- Rotation @ a Fixed Axis

Using the cross-product relationship of vector notation:

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \qquad \mathbf{r} \times \boldsymbol{\omega} = -\mathbf{v}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

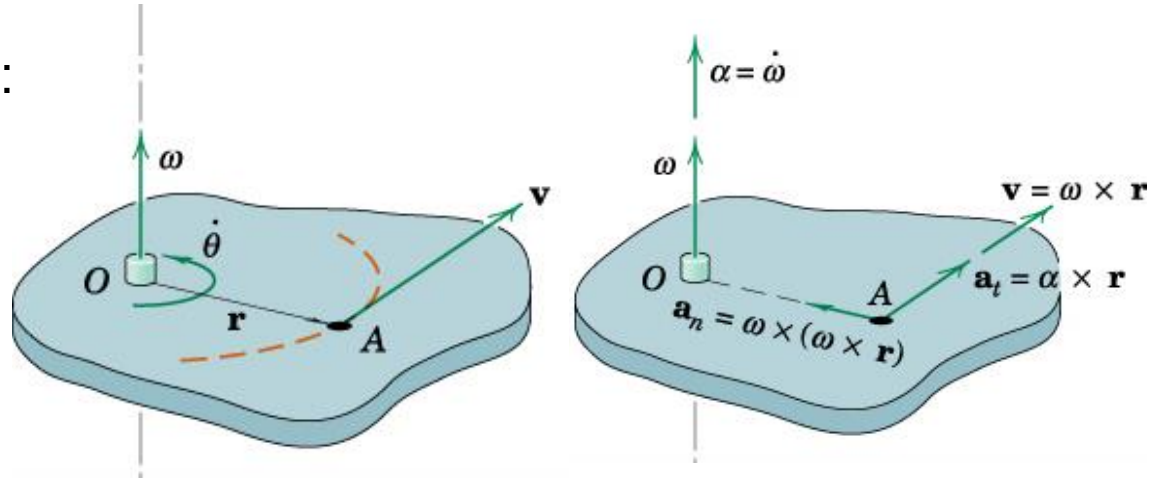
$$= \boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\alpha} \times \mathbf{r}$$

The vector equivalent eqns:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$$



Plane Kinematics of Rigid Bodies

Absolute Motion Analysis

•First approach to rigid body kinematics

•Similar to the absolute motion analysis of the constrained motion of connected particles discussed in kinematics of particles (for pulley configurations considered, the relevant velocities and accelerations were obtained by successive differentiation of the lengths of the connecting cables).

•In case of absolute motion analysis of rigid bodies, both linear and angular velocities and linear and angular accelerations are required to be considered.

Plane Kinematics of Rigid Bodies

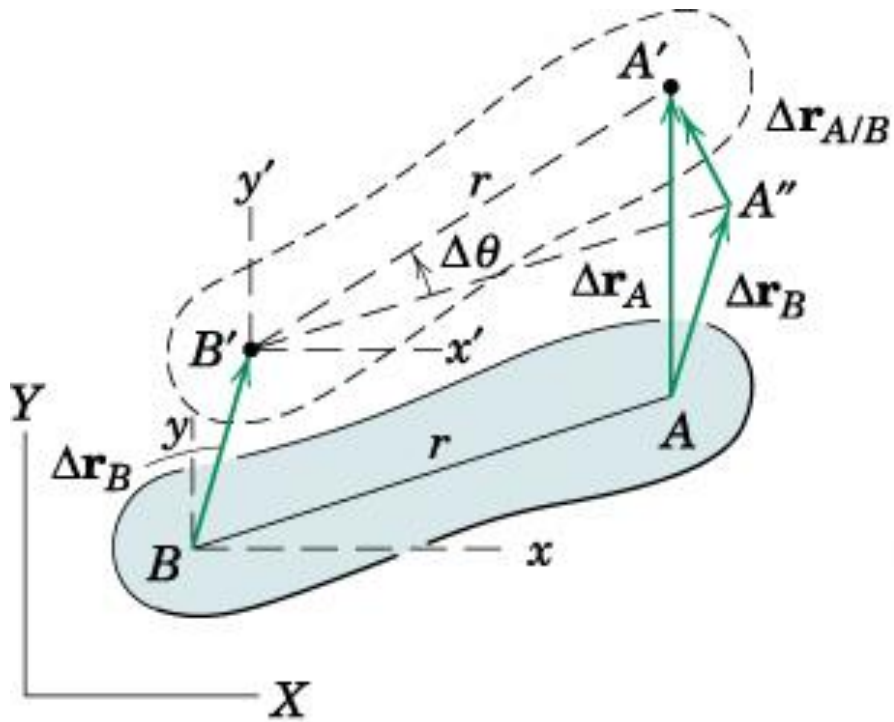
Relative Motion Analysis: Velocity

- Second approach to rigid body kinematics

- In case of kinematics of particles, motion relative to translating axes:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Relative velocity due to Rotation: *AB* → *A'B'* during Δt $\quad \Delta\mathbf{r}_{B/A} = -\Delta\mathbf{r}_{A/B}$

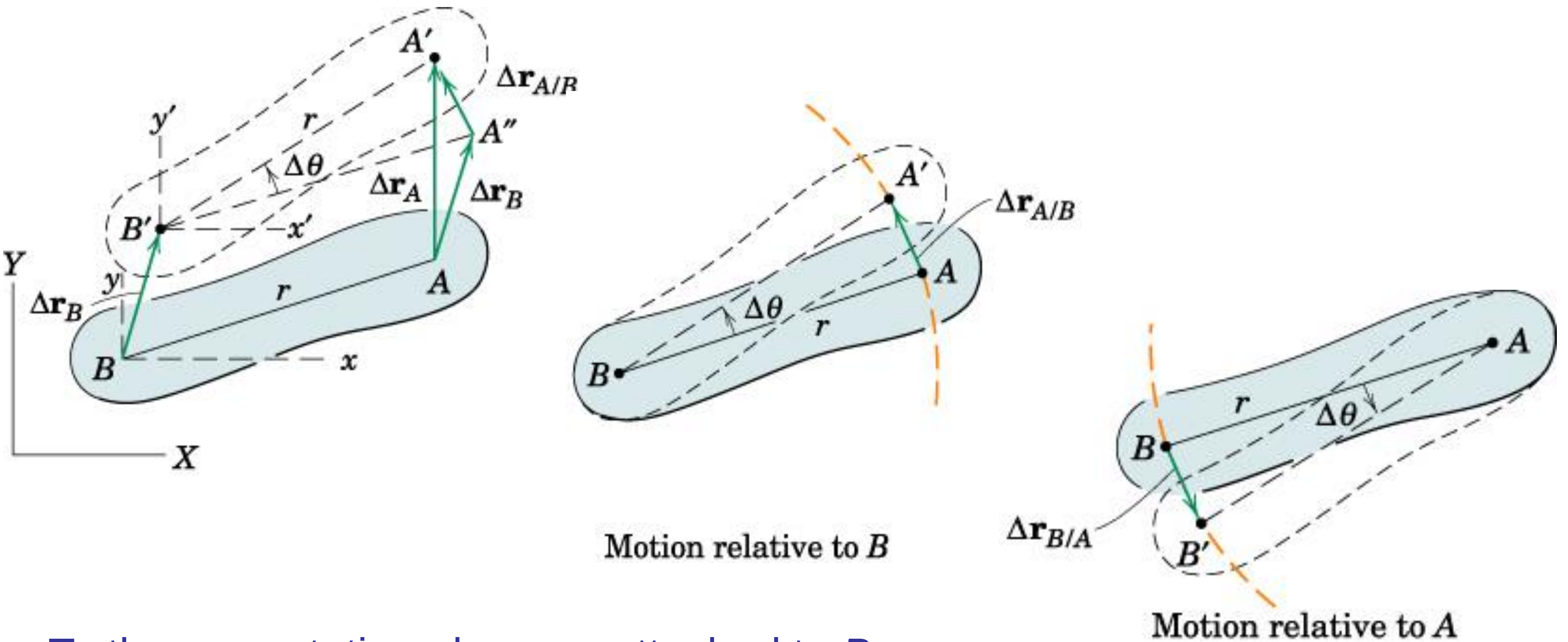


- First translation, then rotation of the body

x'-y' :: non-rotating reference axes

Plane Kinematics of Rigid Bodies

- Relative Motion Analysis: Velocity



To the non-rotating observer attached to *B*,

:: the body appears to undergo fixed axis rotation @ *B*

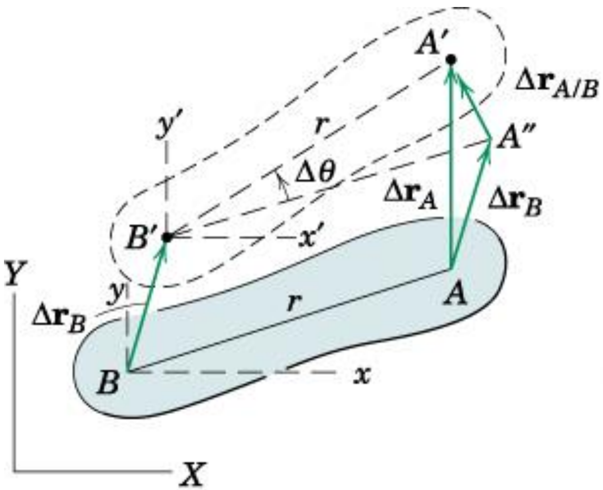
:: **Point *A*** executes **circular motion**

:: **Equations** developed for **circular motions** can be used

Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity

Relative velocity due to Rotation



$\Delta \mathbf{r}_A = \Delta \mathbf{r}_B + \Delta \mathbf{r}_{A/B}$

$\Delta \mathbf{r}_{A/B}$ = has the magnitude $r\Delta\theta$ as $\Delta\theta$ approaches zero

•$\Delta \mathbf{r}_{A/B}$ is accompanied by $\Delta\theta$ as seen from translating axes *x'*-*y'*

Relative velocity Equation → $\boxed{\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}}$

Magnitude of the relative vel:

$$v_{A/B} = \lim_{\Delta t \to 0} (|\Delta \mathbf{r}_{A/B}|/\Delta t) = \lim_{\Delta t \to 0} (r\Delta\theta/\Delta t) \qquad \omega = \dot{\theta} \qquad \rightarrow \boxed{v_{A/B} = r\omega}$$

Using **r** to represent vector $\mathbf{r}_{A/B}$, relative velocity may be written as the vector:

From already derived vector eqn for velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

$\boxed{\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}}$ $\boldsymbol{\omega}$ is the angular velocity vector normal to the plane of the motion

Relative linear velocity is always perpendicular to the line joining two points in question (*A*-*B* or *A'*-*B*)

Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity

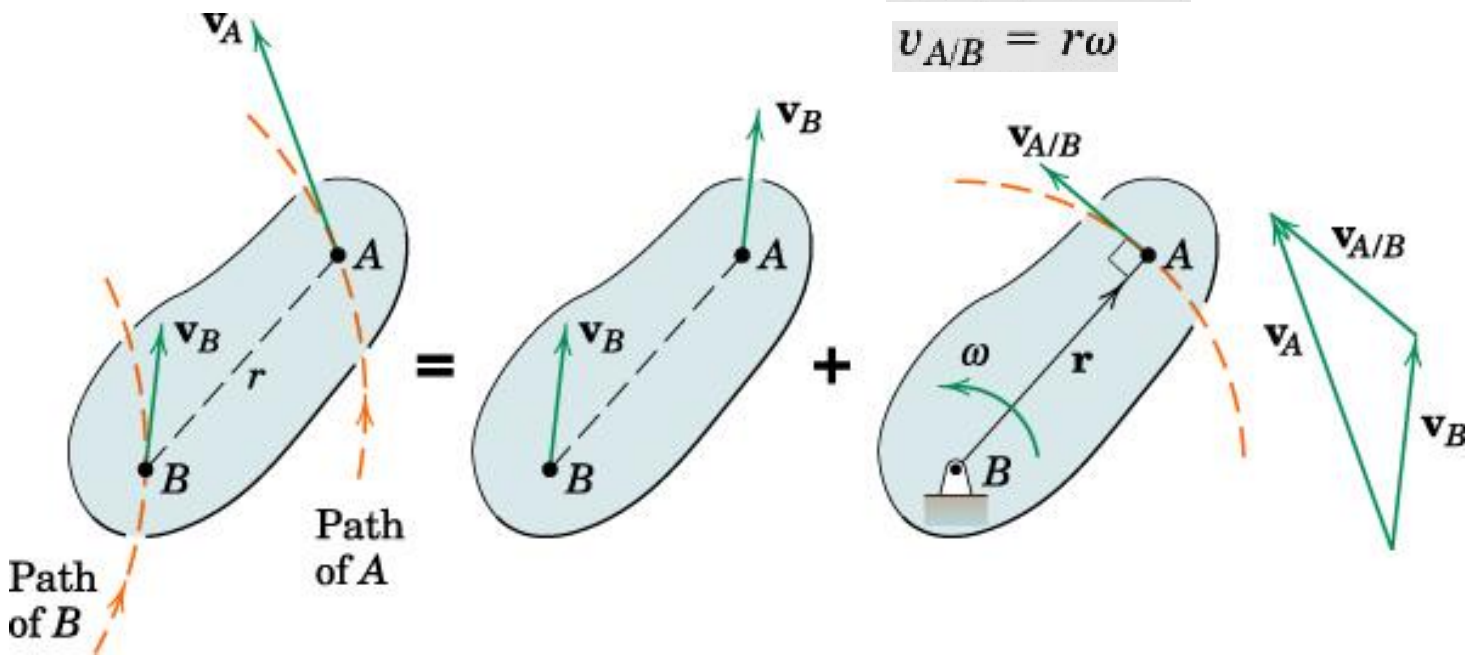
Relative velocity due to Rotation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

- Visualizing the separate translation and rotation components of the relative velocity equation
- Choosing *B* as the reference point
- Relative linear velocity is always perpendicular to the line joining two points in question (*A*-*B*)
- ω is the absolute angular velocity of *AB*.

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$$

$$v_{A/B} = r\omega$$

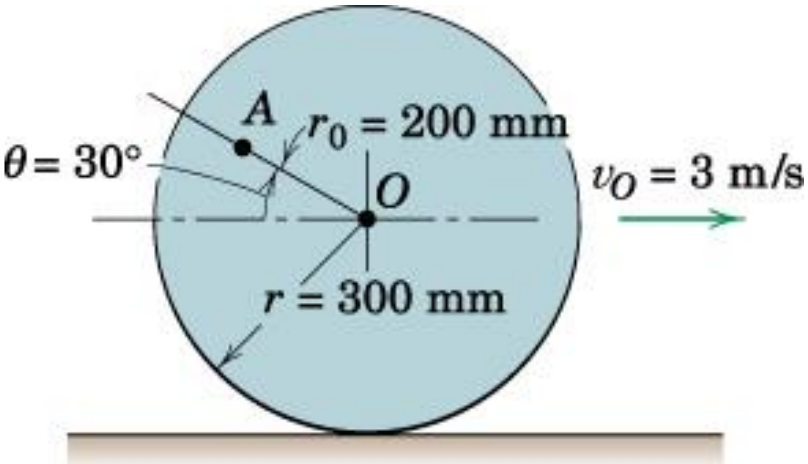


Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity

Example

The wheel of radius $r = 300$ mm rolls to the right without slipping and has a velocity $v_O = 3$ m/s of its center O. Calculate the velocity of point A on the wheel for the instant represented.



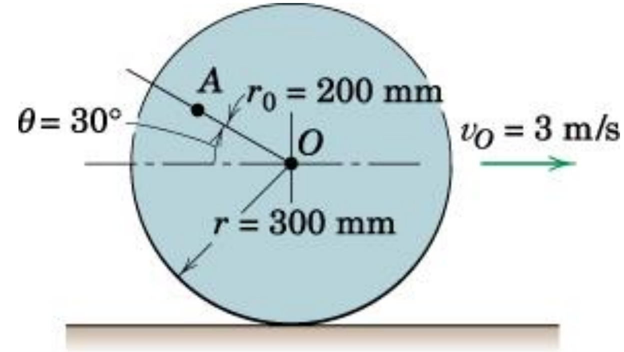
Solution: Choosing center *O* as the reference point for relative velocity eqn since its motion is given.

$$\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O}$$

The relative velocity of *A* wrt *O* is observed from the translating axes *x-y* attached to *O*.

Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity



Example

Solution: $\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A/O}$

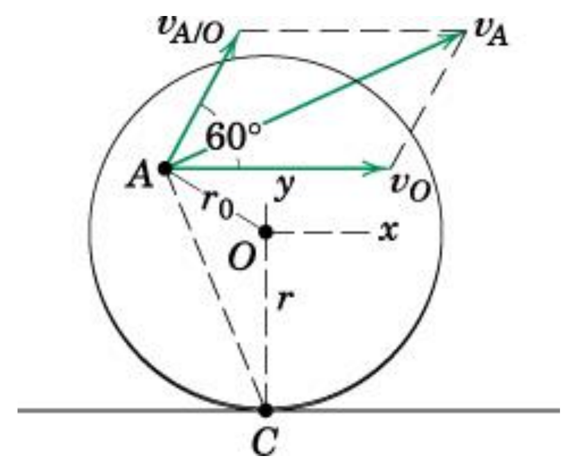
Direction of relative velocity is perpendicular to *OA*. The angular velocity of *OA* is the same as that of the wheel: $\omega = v_O/r = 3/0.3 = 10$ rad/s

$v_{A/B} = r\omega$ $\quad$ $v_{A/O} = 0.2(10) = 2$ m/s

The vector sum $\mathbf{v}_A$ can be calculated from law of cosines:

$$v_A{}^2 = 3^2 + 2^2 + 2(3)(2)\cos 60° = 19\ (\text{m/s})^2$$

$v_A = 4.36$ m/s $\quad$ Dirn of v_A can be easily found out.



Point C momentarily has zero velocity and may also be chosen as a reference point: $\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{A/C} = \mathbf{v}_{A/C}$

$$v_{A/C} = \overline{AC}\omega = \frac{\overline{AC}}{\overline{OC}} v_O = \frac{0.436}{0.300}(3) = 4.36 \text{ m/s}$$

AC = 200sin30 + [(200cos30)^2+(300)^2]
= 436 mm

$v_A = v_{A/C} = 4.36$ m/s

The problem may also be solved using vector formulation.

Plane Kinematics of Rigid Bodies

Instantaneous Center of Zero Velocity

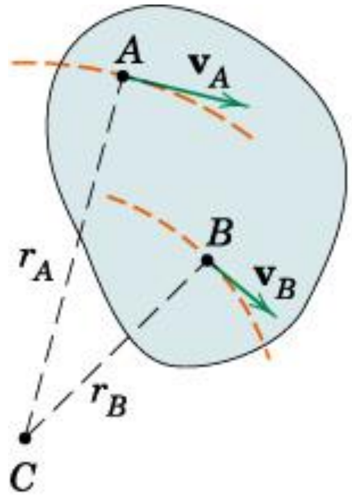
•Relative Motion Analysis: velocity of a point on a rigid body in plane motion = relative velocity due to rotation @ a convenient reference point + velocity of the reference point.

•The problem can also be solved by choosing a unique reference point that momentarily has zero velocity.

- This point lies on **Instantaneous Axis of Zero Velocity** (or instantaneous axis of rotation), and **intersection** of this **axis** with the **plane of motion** is known as **Instantaneous Center of Zero Velocity**.
- As far as velocities are concerned, the **body** may be considered to be in **pure rotation** in a **circular path** @ the Instantaneous Axis of Zero Velocity.
- Locating the instantaneous center of zero velocity is important to simplify the solution of many problems involving rigid body rotations

Plane Kinematics of Rigid Bodies

Instantaneous Center of Zero Velocity



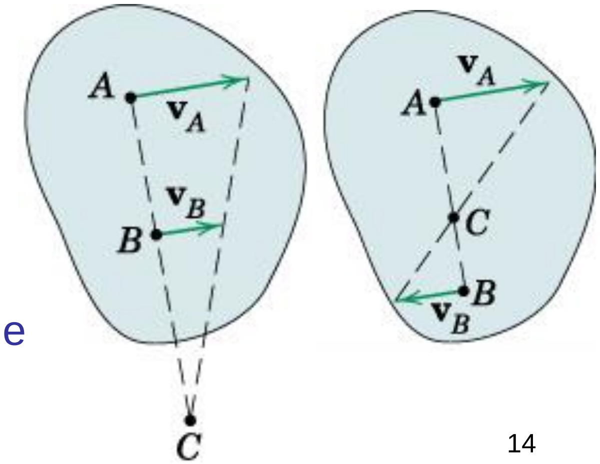
Locating the Instantaneous Center

- Assume that the dirns of absolute vel of any points *A* and *B* on rigid body are known and are not parallel.
- If *C* is a point @ which *A* has absolute circular motion at the instant considered, *C* must lie on the normal to $\mathbf{v}_A$ through *A*.
- Similarly, point *B* can also have absolute circular motion @ *C* at the instant considered.

→ Intersection of two perpendiculars will give the absolute center of rotation at the instant considered.

→ Point C is the instantaneous center of zero velocity and may lie on or off the body

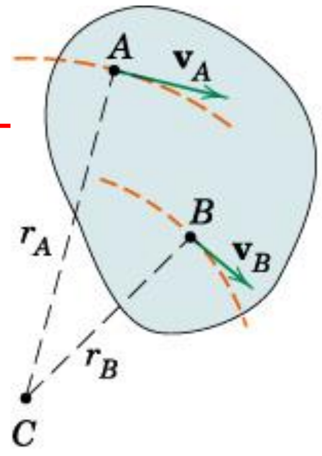
→ Instantaneous center of zero velocity need not be a fixed point in the body or a fixed point in the plane.



If vel of two points are parallel and the line joining the points is perpendicular to the dirn of vel, the instantaneous center is located by direct proportion.

As the parallel velocities become equal in magnitude, the IC moves farther away from the body and approaches infinity in the limit → body stops rotating and translates only.

Plane Kinematics of Rigid Bodies



Instantaneous Center of Zero Velocity

Locating the Instantaneous Center

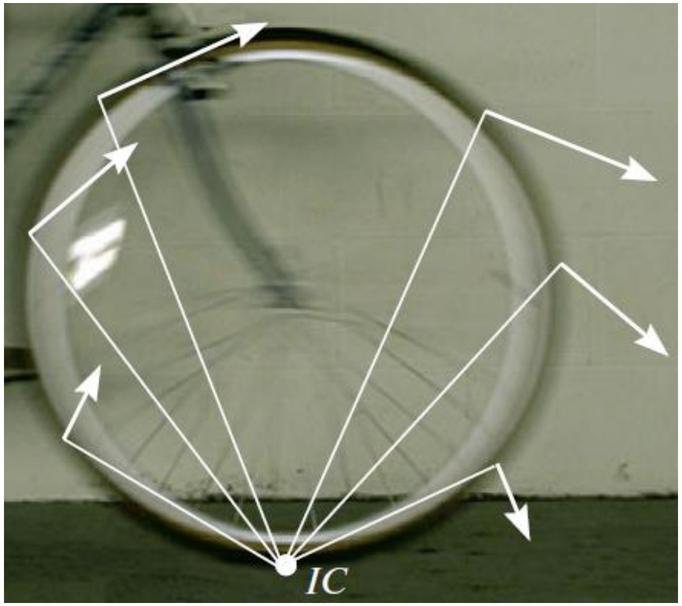
If magnitude of velocity at one of the points on the rigid body under general plane motion is known (v_A)

→ Angular velocity of the body ω and linear velocity of every point in the body can be easily obtained using:

$\omega = v_A / r_A$ → this is also the angular velocity of every line in the body

→ velocity of B is $v_B = r_B\,\omega = (r_B / r_A)\, v_A$

- Once the IC is located, the dirn of the instantaneous velocity of every point in the body is readily found since it must be perpendicular to the radial line joining the point in question with IC.
  Velocity of all such points: $v = r\,\omega$



Note that the spokes are somewhat visible near IC, whereas at the top of the wheel they become blurred. Also note how the direction of velocity varies at different point on the rim of the wheel.

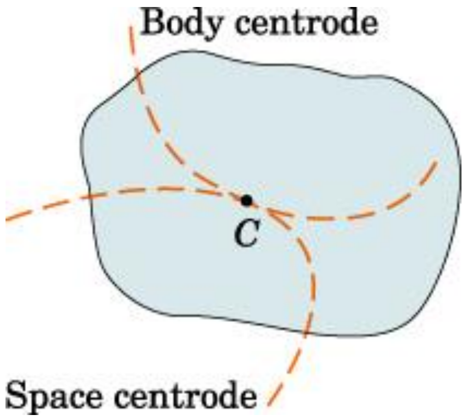
Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity

Instantaneous Center of Zero Velocity

Motion of the Instantaneous Center

- IC changes its position in space and on the body as body changes its position
- Locus of the ICs in space is known as space centrode
- Locus of the ICs on the body is known as body centrode
- At the instant considered, the two curves are tangent at C
- The body centrode rolls on the space centrode during the motion of the body.



- Although the IC of zero velocity is momentarily at rest, its accln is generally not zero.
  → IC cannot be used as an IC of zero acceleration.
- Instantaneous center of zero acceleration exist for bodies in general plane motion → will not be discussed in ME101

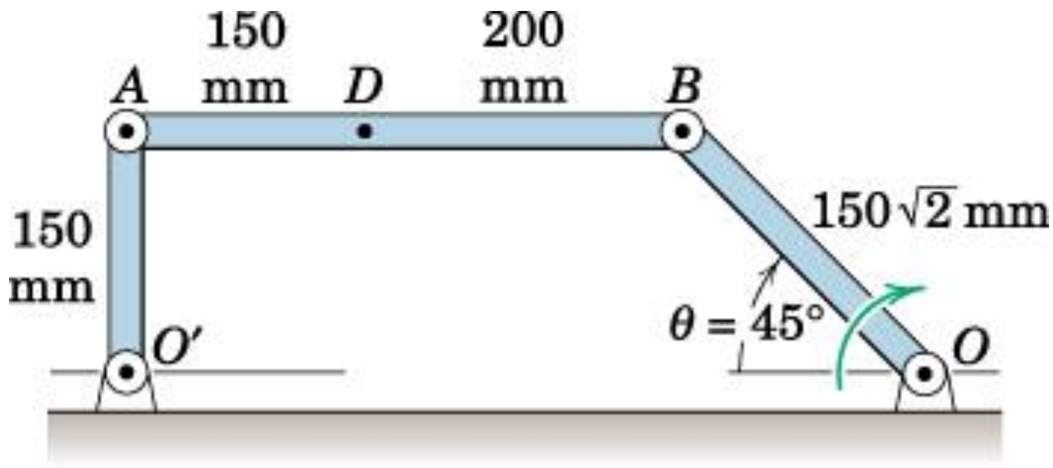
Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Velocity

Instantaneous Center of Zero Velocity

Example

Arm OB of the linkage has a clockwise angular velocity of 10 rad/sec in the position shown where $\theta = 45°$. Determine the velocity of A, the velocity of D, and the angular velocity of link AB for the position shown.



Plane Kinematics of Rigid Bodies

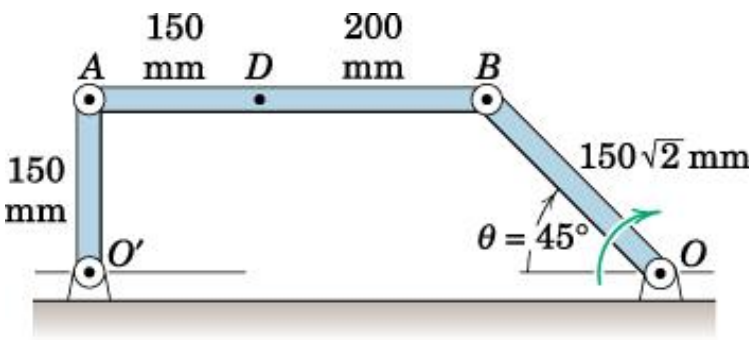
Instantaneous Center of Zero Velocity

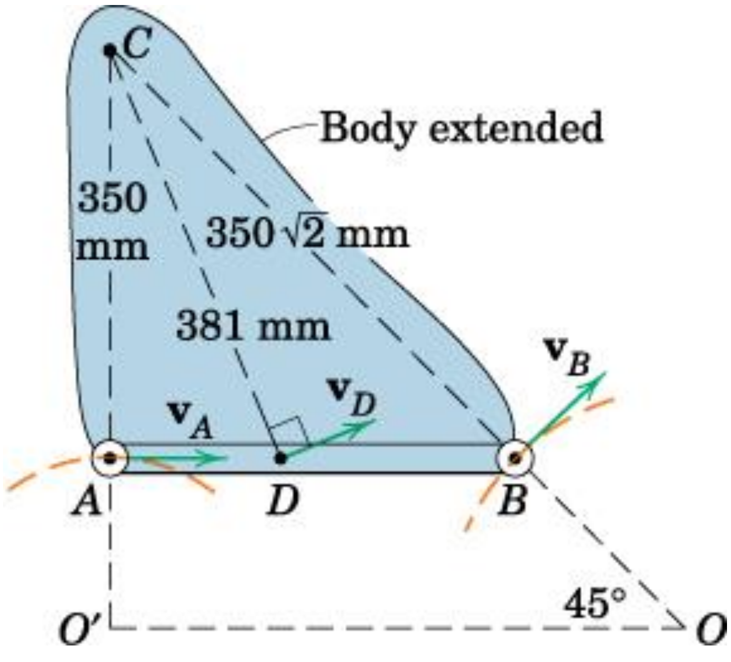
Example

Solution:

Directions of vel of *A* and *B* are tangent to their circular paths @ fixed centers *O* and *O'*

- Intersection of the two perpendiculars to the vel from A and B → IC *C* for the link *AB*





$AC = AB\tan 45 = 350\tan 45 = 350$ mm

$CD^2 = 350^2 + 150^2 \rightarrow CD = 381$ mm

$CB^2 = 350^2 + 350^2 \rightarrow CB = 350\sqrt{2}$ mm

Angular vel of *OB*: $\omega_{OB} = 10$ rad/s

$v_B = r\,\omega = 150\sqrt{2} \times 10 = 2121.3$ mm/s

Ang vel of *BC*: $\omega_{BC} = 2121.3/350\sqrt{2} = 4.29$ rad/s

BC is a line considered on the extended rigid body → $\omega_{BC} = \omega_{AC} = \omega_{DC} = \omega_{AB}$ (clockwise)

$v_A = r\,\omega = 350 \times 4.29 = 1.5$ m/s

$v_D = r\,\omega = 381 \times 4.29 = 1.632$ m/s

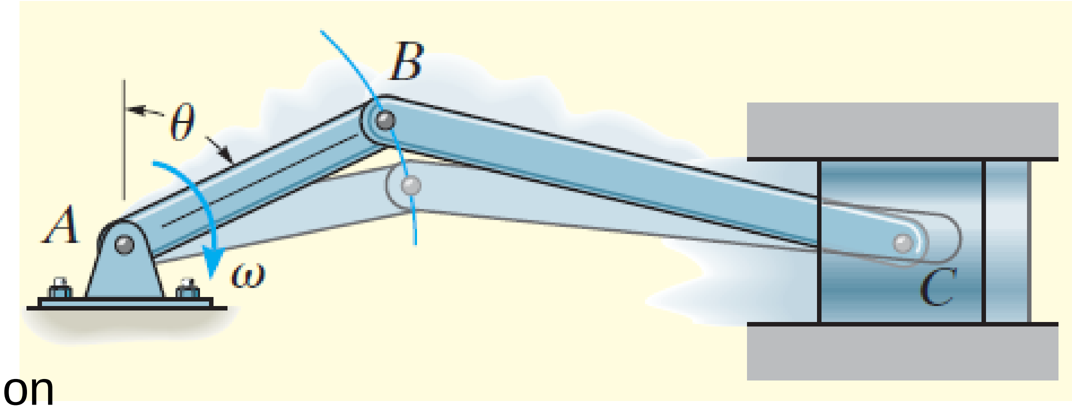
Directions of velocities are shown.

Plane Kinematics of Rigid Bodies

Instantaneous Center of Zero Velocity

Example

Show the instantaneous center of zero velocity for member *BC*.



Solution:

Point *B* follow circular path of motion because link *AB* is subjected to **rotation about a fixed axis**.

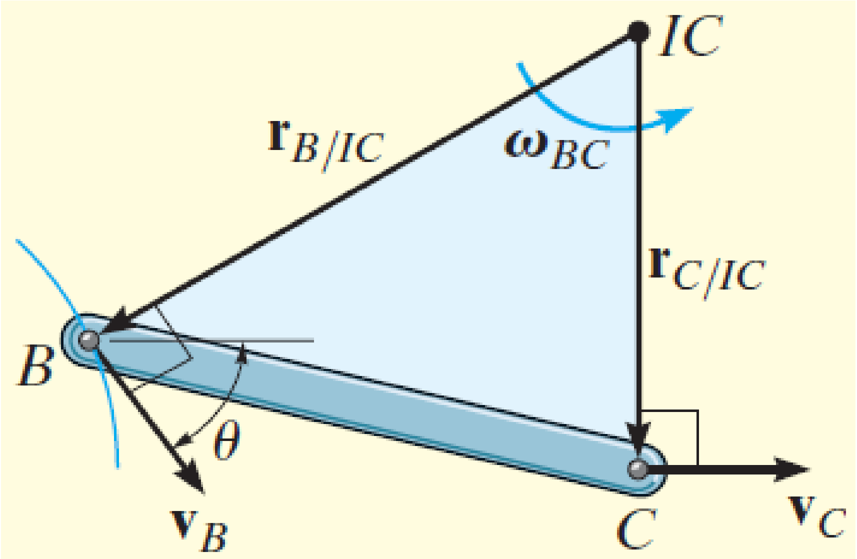
→ $\mathbf{v}_B$ is perpendicular to *AB*

Motion of *B* causes the piston to move horizontally.

→ $\mathbf{v}_C$ is horizontal

Draw lines perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$

→ These lines intersect at IC.

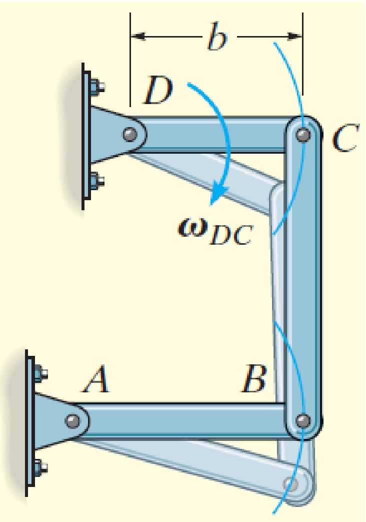


Plane Kinematics of Rigid Bodies

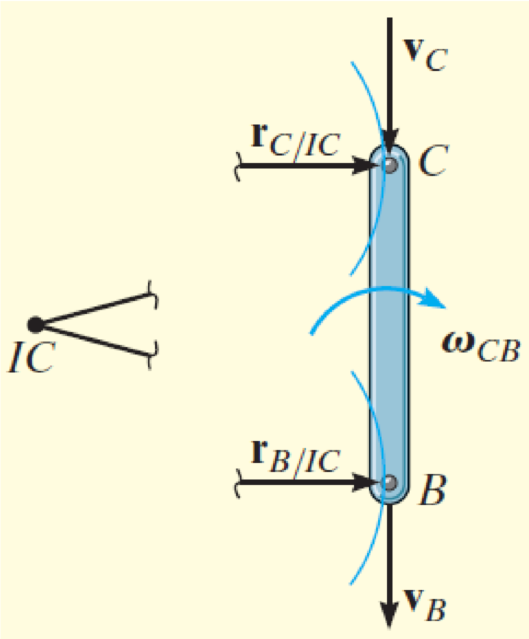
Instantaneous Center of Zero Velocity

Example

Show the instantaneous center of zero velocity for the link *CB*.



Solution:



Points *B* and *C* follow circular path of motion because links *AB* and *DC* are subjected to **rotation about a fixed axis**.

→ $\mathbf{v}_B$ and $\mathbf{v}_C$ are perpendicular to *AB* and *DC* at the instant considered

Radial lines drawn perpendicular to $\mathbf{v}_B$ and $\mathbf{v}_C$ are parallel and Intersect at infinity

→ $\omega_{CB} = \mathbf{v}_C / \infty = 0$

→ Link *CB* momentarily translates

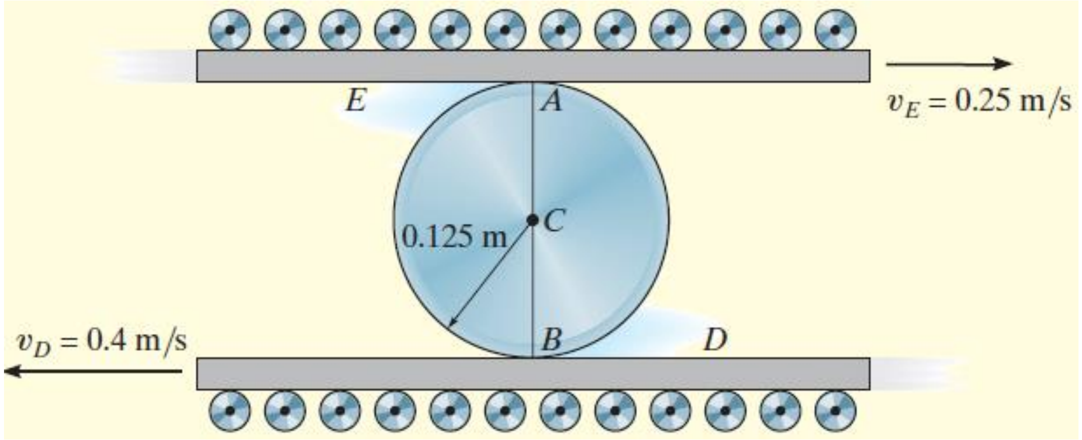
→ An instant later, *CB* will move to a tilted position, causing IC to move to some finite location.

Plane Kinematics of Rigid Bodies

Instantaneous Center of Zero Velocity

Example

The cylinder rolls without slipping betn the two moving plates. Determine the location of IC and calculate the angular velocity of the cylinder and velocity of its center *C*.

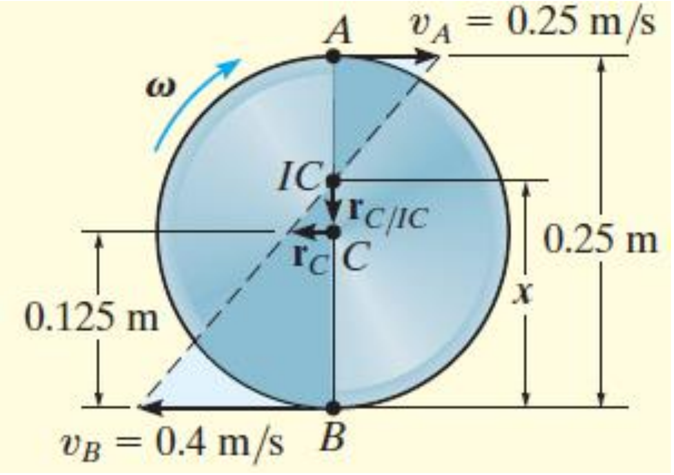


Solution:

Since no slipping occurs, velocities of points *A* and *B* on cylinder will be same as v_E and v_D resp.

Since v_A and v_B are parallel, IC can be located by proportionality of right triangles **on line joining *AB***.

Direction of velocity of *C* and hence that of angular velocity of cylinder can also be known from the similar triangles.



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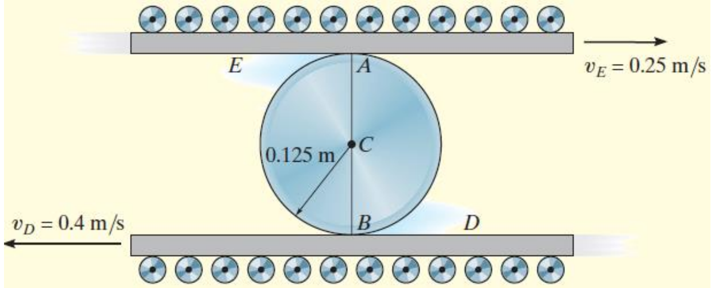
Plane Kinematics of Rigid Bodies

Instantaneous Center of Zero Velocity



Example: Solution:

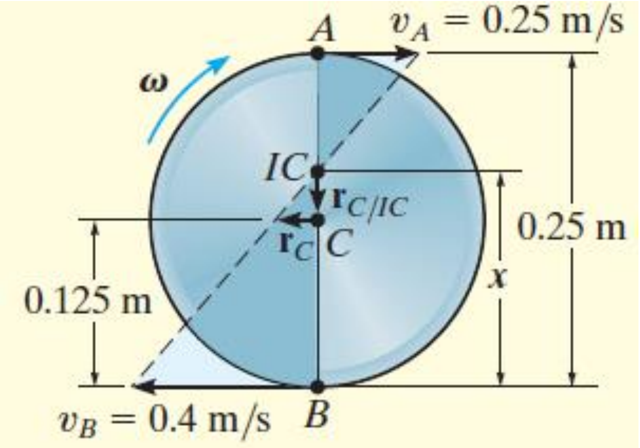
$$v_B = \omega x; \qquad 0.4 \text{ m/s} = \omega x$$

$$v_A = \omega(0.25 \text{ m} - x); \qquad 0.25 \text{ m/s} = \omega(0.25 \text{ m} - x)$$

Dividing one eqn by other to eliminate ω:

$$0.4(0.25 - x) = 0.25x$$

$$x = \frac{0.1}{0.65} = 0.1538 \text{ m}$$



$$\omega = \frac{v_B}{x} = \frac{0.4 \text{ m/s}}{0.1538 \text{ m}} = 2.60 \text{ rad/s} \circlearrowright$$

$$v_C = \omega r_{C/IC} = 2.60 \text{ rad/s}\,(0.1538 \text{ m} - 0.125 \text{ m})$$

$$= 0.0750 \text{ m/s} \leftarrow$$

Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Acceleration

Eqn that describes the relative velocities of two points *A* and *B* in plane motion in terms of nonrotating reference axes:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Differentiating wrt time:

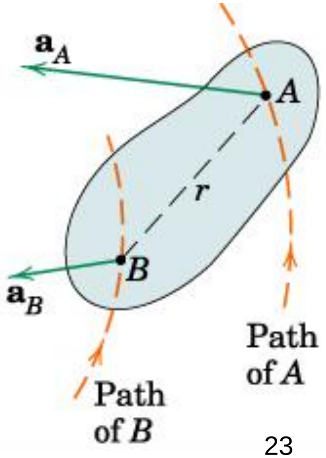
$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

→ Acceleration of point *A* is equal to vector sum of acceleration of point *B* and the acceleration which *A* appears to have to a nonrotating observer moving with *B*.

Relative Acceleration due to Rotation

Consider two points on a rigid body. Since distance between them remains the same, the observer moving with *B* perceives *A* to have circular motion about *B*.

- Since the relative motion is circular, relative acceleration term will have both normal as well as tangential components.
- Normal component of accln will be directed from *A* towards B due to change in direction of $\mathbf{v}_{A/B}$.
- Tangential component of accln will be perpendicular to *AB* due to the change in the magnitude of $\mathbf{v}_{A/B}$.



$\mathbf{a}_A$ and $\mathbf{a}_B$ are the absolute accelerations of *A* and *B*.

→ Not tangent to the path of motion when the motion is curvilinear.

Plane Kinematics of Rigid Bodies

Relative Motion Analysis: Acceleration

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

We can write:

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t$$

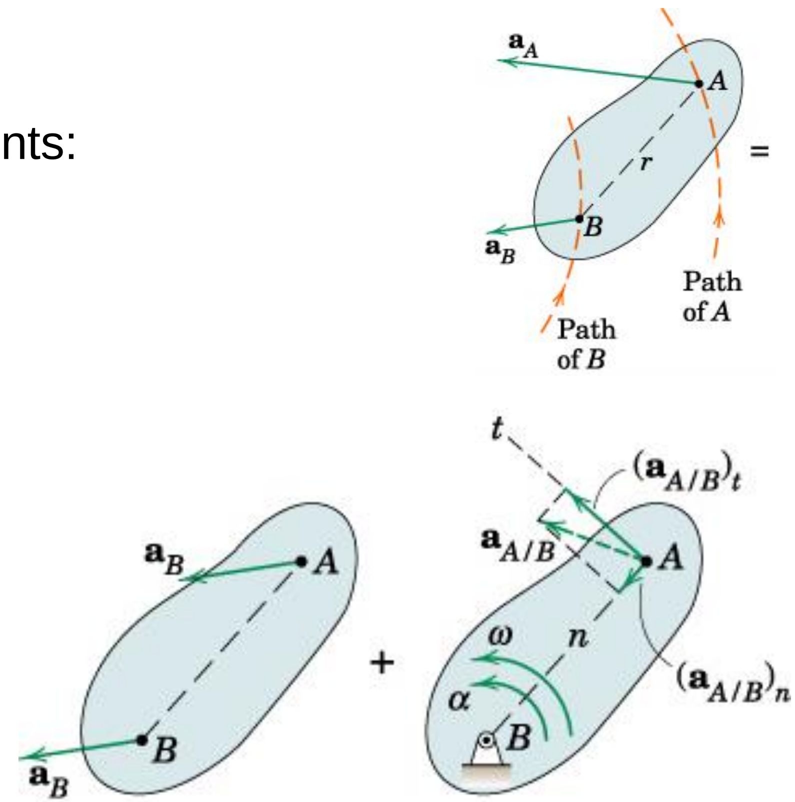
The magnitudes of the relative accln components:

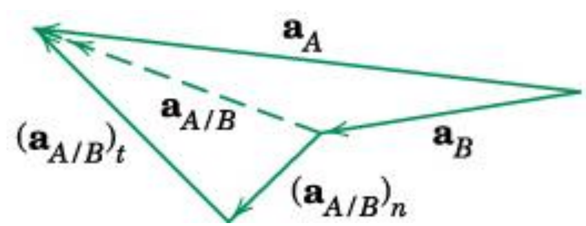
$$(a_{A/B})_n = {v_{A/B}}^2/r = r\omega^2$$
$$(a_{A/B})_t = \dot{v}_{A/B} = r\alpha$$

Acceleration components in vector notations:

$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$
$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \times \mathbf{r}$$

ω is the angular velocity of the body
α is the angular acceleration of the body
r is the vector locating *A* from *B*
→ The relative accln terms depend on the respective absolute angular vel and absolute angular accln.





Alternatively: $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$