Virtual Work

Elastic Potential Energy (V_e)

Consider a **linear and elastic** spring compressed by a force F → $F = kx$

k = spring constant or stiffness of the spring

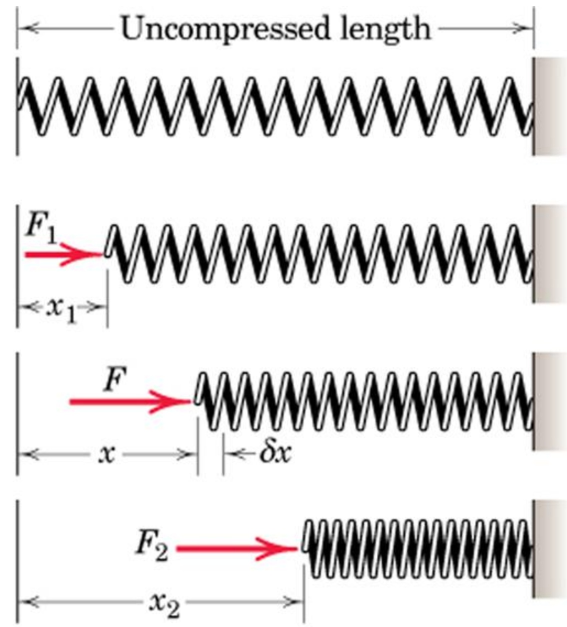
Work done on the spring by F during a movement dx: $dU = F\,dx$

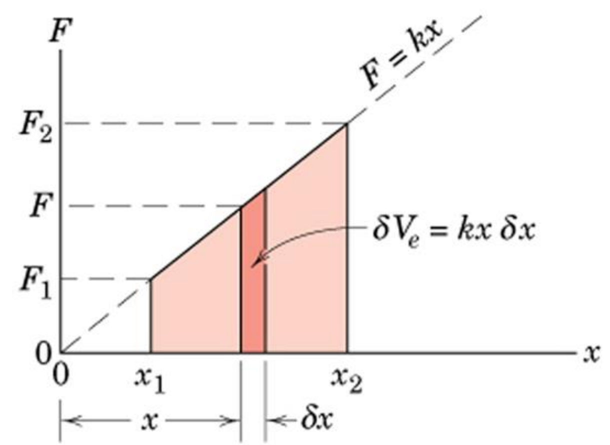
Elastic potential energy of the spring for compression x = total work done on the spring

$$V_e = \int_0^x F\,dx = \int_0^x kx\,dx$$

$$\boxed{V_e = \tfrac{1}{2}kx^2}$$

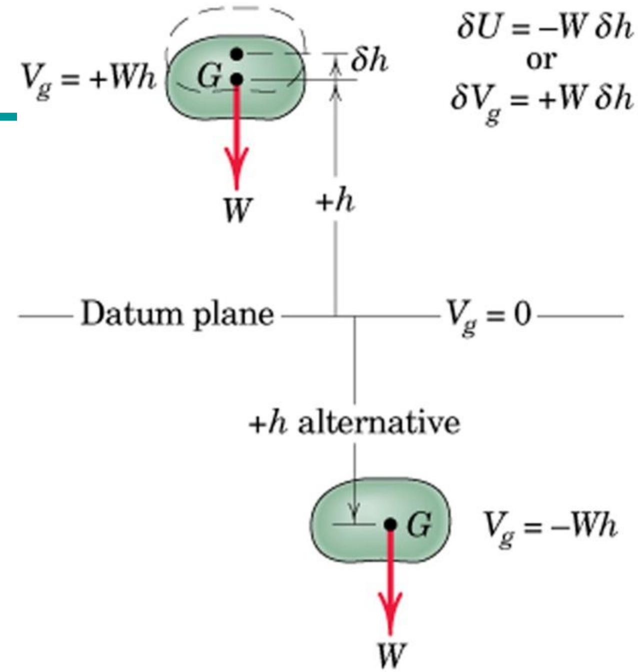
→ Potential Energy of the spring = area of the triangle in the diagram of F versus x from 0 to x





Virtual Work

Gravitational Potential Energy (V_g)



Treatment of Gravitational force till now:

For an upward displacement δh of the body, work done by the weight (W=mg) is: $\delta U = -mg\delta h$

For downward displacement (with h measured positive downward): $\delta U = mg\delta h$

Alternatively, work done by gravity can be expressed in terms of a change in potential energy of the body.

The Gravitational Potential Energy of a body is defined as the work done on the body by a force **equal and opposite to the weight** in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to be zero. → V_g is negative of the work done by the weight.

V_g =0 at h=0

→ at height h above the datum plane, V_g of the body = mgh

→ If the body is at distance h below the datum plane, V_g of the body = -mgh

Virtual change in the gravitational potential energy: $\boldsymbol{\delta V_g = +mg\delta h}$

δh is the upward virtual displacement of the mass centre of the body

If mass centre has downward virtual displacement → $\boldsymbol{\delta V_g = -mg\delta h}$

Virtual Work :: Energy Equation

Work done by the linear spring *on the body to which the spring is attached (**during displacement of the spring**) is the **negative of the change in the elastic potential energy of the spring**.*

Work done by the gravitational force *or **weight** is the **negative of the change in gravitational potential energy***

Virtual Work eqn to a system with springs and with changes in the vertical position of its members → **replace** the **work on springs** and the **work by weights** by **negative of the respective potential energy changes**

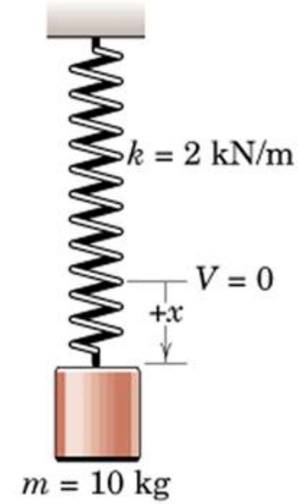
Total Virtual Work δU = work done by all active forces ($\delta U'$) other than spring forces and weight forces + the work done by the spring forces and weight forces, i.e., $-(\delta V_e + \delta V_g)$

$$\boxed{\delta U' - (\delta V_e + \delta V_g) = 0} \quad \text{or} \quad \boxed{\delta U' = \delta V}$$

$V = V_e + V_g$ → Total Potential Energy of the system

Virtual Work :: Energy Equation

Stability of Equilibrium



Consider a system where movement is accompanied by changes in gravitational and elastic potential energy:
If work done by all active forces other than spring forces and weight forces is zero → *δU' = 0* (No work is done on the system by the non-potential forces)

→ $\boxed{\delta(V_e + V_g) = 0}$ or $\boxed{\delta V = 0}$

→Equilibrium configuration of a mechanical system is one for which the total potential energy *V* of the system has a **stationary value**.

For a SDOF system (where **potential energy** and its **derivatives are continuous functions of a single variable**, *x* that **describes the configuration**), it is equivalent to state that:

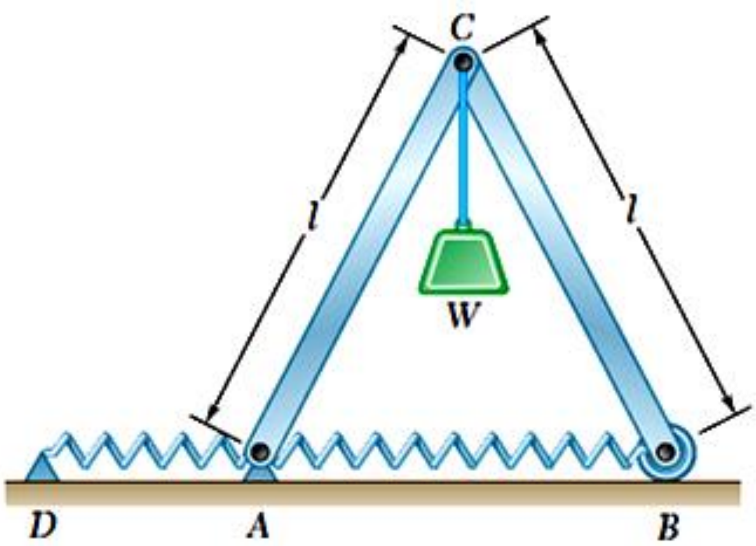
$$\boxed{\frac{dV}{dx} = 0}$$

→ A mechanical system is in equilibrium when the derivative of its total potential energy is zero

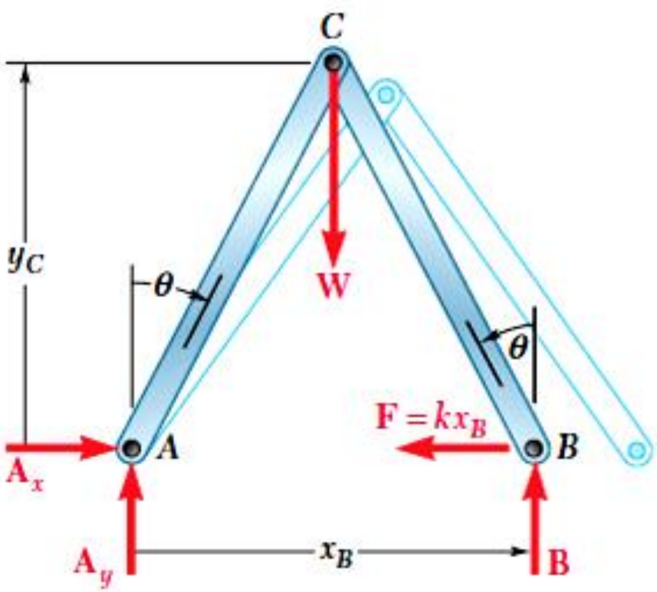
→For systems with **multiple DOF**, **partial derivative of *V* wrt each coordinate must be zero for equilibrium**.

Example on Potential

A two-membered structure is supporting a weight W. The original length of the spring is AD. Neglecting the friction forces and the weight of the members obtain the possible equilibrium configurations.



Example on Potential



:: The only forces contributing to the work done are: (a) W and (b) spring force F

:: Total potential energy = Gravitational potential energy + Elastic potential energy

Choosing a coordinate system with origin at A and noting that the deflection of the spring, measured from its undeformed position, is $AB = x_B$, we write

$$V_e = \tfrac{1}{2}kx_B^2 \qquad V_g = Wy_C$$

Example on Potential

Expressing the coordinates x_B and y_C in terms of the angle θ, we have

$$x_B = 2l \sin\theta \qquad y_C = l\cos\theta$$
$$V_e = \tfrac{1}{2}k(2l \sin\theta)^2 \qquad V_g = W(l\cos\theta)$$
$$V = V_e + V_g = 2kl^2 \sin^2\theta + Wl\cos\theta$$

The positions of equilibrium of the system are obtained by equating to zero the derivative of the potential energy V. We write

$$\frac{dV}{d\theta} = 4kl^2 \sin\theta\cos\theta - Wl\sin\theta = 0$$

or, factoring $l \sin\theta$,

$$\frac{dV}{d\theta} = l\sin\theta(4kl\cos\theta - W) = 0$$

There are therefore two positions of equilibrium, corresponding to the values $\theta = 0$ and $\theta = \cos^{-1}(W/4kl)$, respectively.

Virtual Work

$$\frac{dV}{dx} = 0$$

Stability of Equilibrium: SDOF

Three Conditions under which this eqn applies.
when total potential energy is:

→ A Minimum (Stable Equilibrium)
→ A Maximum (Unstable Equilibrium)
→ A Constant (Neutral Equilibrium)

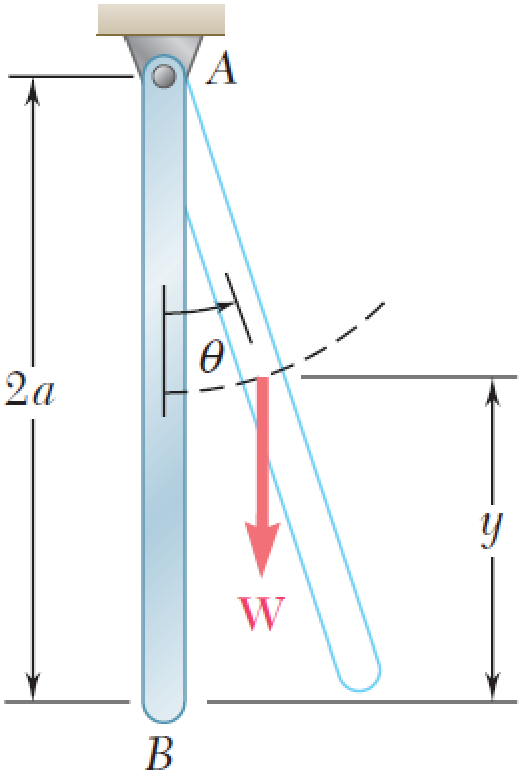
Total Potential Energy: Minimum Maximum Constant

A small displacement away from the STABLE position results in an increase in potential energy and a tendency to return to the position of lower energy.

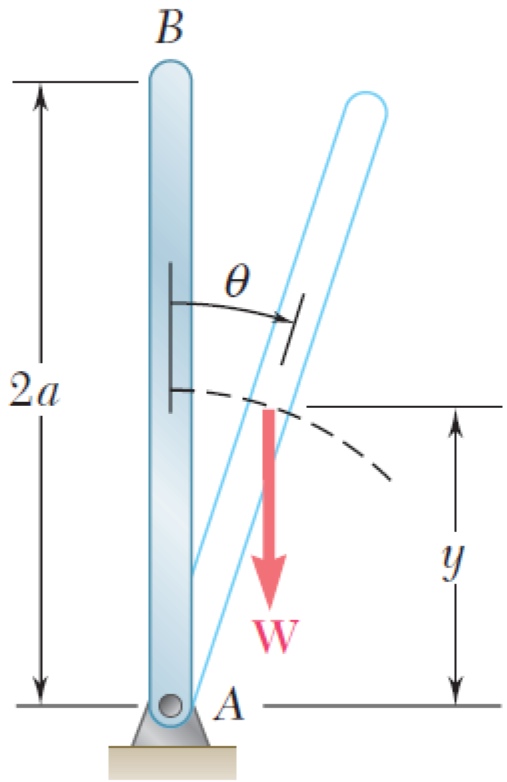
A small displacement away from the UNSTABLE position results in a decrease in potential energy and a tendency to move farther away from the equilibrium position to a position of still lower energy.

For a NEUTRAL position, a small displacement one way or the other results in no change in potential energy and no tendency to move either way.

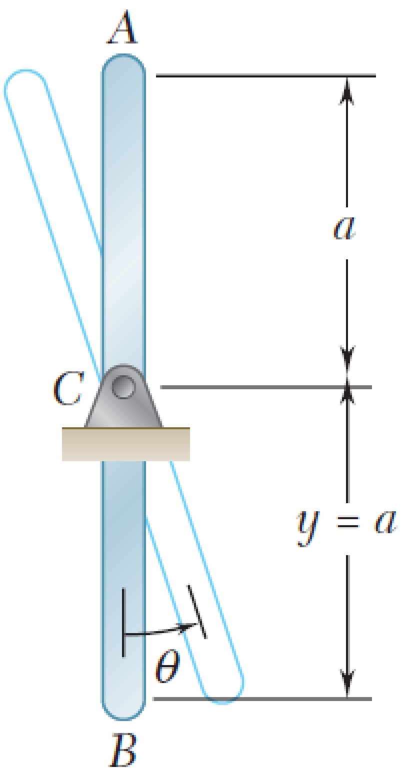
Stability of Equilibrium



Stable Equilibrium



Unstable Equilibrium



Neutral Equilibrium

Stability of Equilibrium

Evaluation for SDOF systems

When a function and its derivatives are continuous:
→the **second derivative** is **positive** **at a point of minimum value of function**
→the **second derivative** is **negative** **at a point of maximum value of function**

Mathematical conditions for equilibrium and stability of a system with a SDOF *x*:

Equilibrium: $\frac{dV}{dx} = 0$

Stable Equilibrium: $\frac{dV}{dx} = 0, \frac{d^2V}{dx^2} > 0$

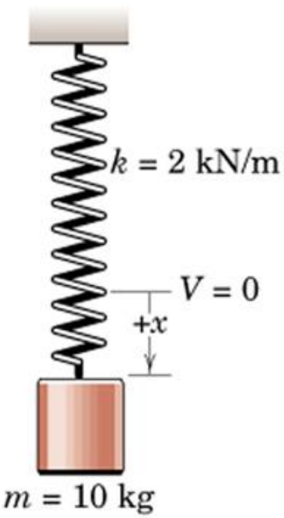
Unstable Equilibrium: $\frac{dV}{dx} = 0, \frac{d^2V}{dx^2} < 0$

Neutral Equilibrium: $\frac{dV}{dx} = \frac{d^2V}{dx^2} = \frac{d^3V}{dx^3} = \ldots = 0$

If $d^2V/dx^2 = 0$ at the equilibrium position, examine the sign of a higher derivative to ascertain the type of equilibrium.

- If the **order** of the **lowest remaining non-zero derivative is even**, the **equilibrium** will be **stable or unstable** according to the **whether the sign of this derivative is positive or negative**
- If the **order** of the **derivative is odd**, the **equilibrium** is classified as **unstable**

Example on Equilibrium



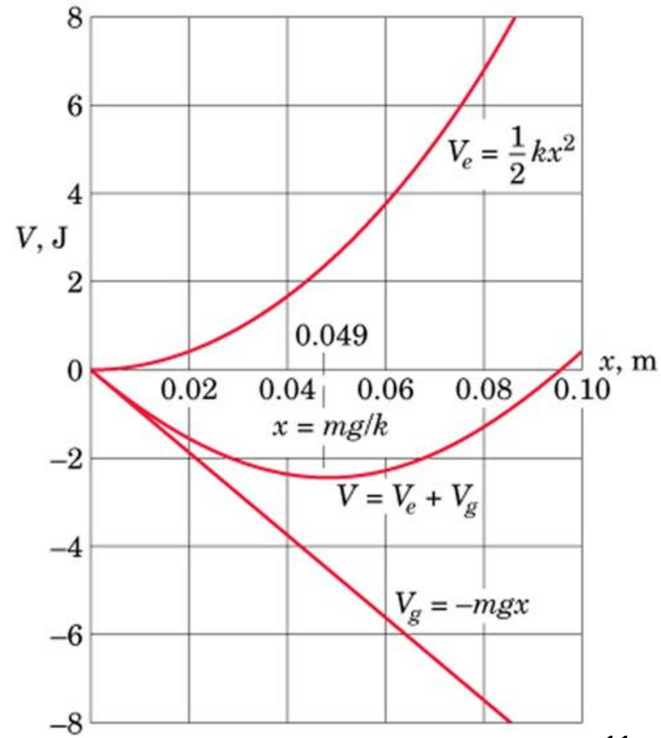
A 10 kg cylinder is suspended by the spring.
Plot the potential energy of the system and show that it is minimum at the equilibrium position.

No external active forces. Choosing datum plane for zero potential energy at the position where the spring is un-extended.
For an arbitrary position x: Elastic potential energy: $V_e = ½ kx^2$
Gravitational Potential Energy: $V_g = - mgx$ (-ve of work done)
Total Potential Energy: $V = V_e + V_g = ½ kx^2 - mgx$
Equilibrium occurs where $\boldsymbol{dV/dx = 0}$
$\rightarrow kx - mg = 0 \rightarrow \boldsymbol{x = mg/k}$

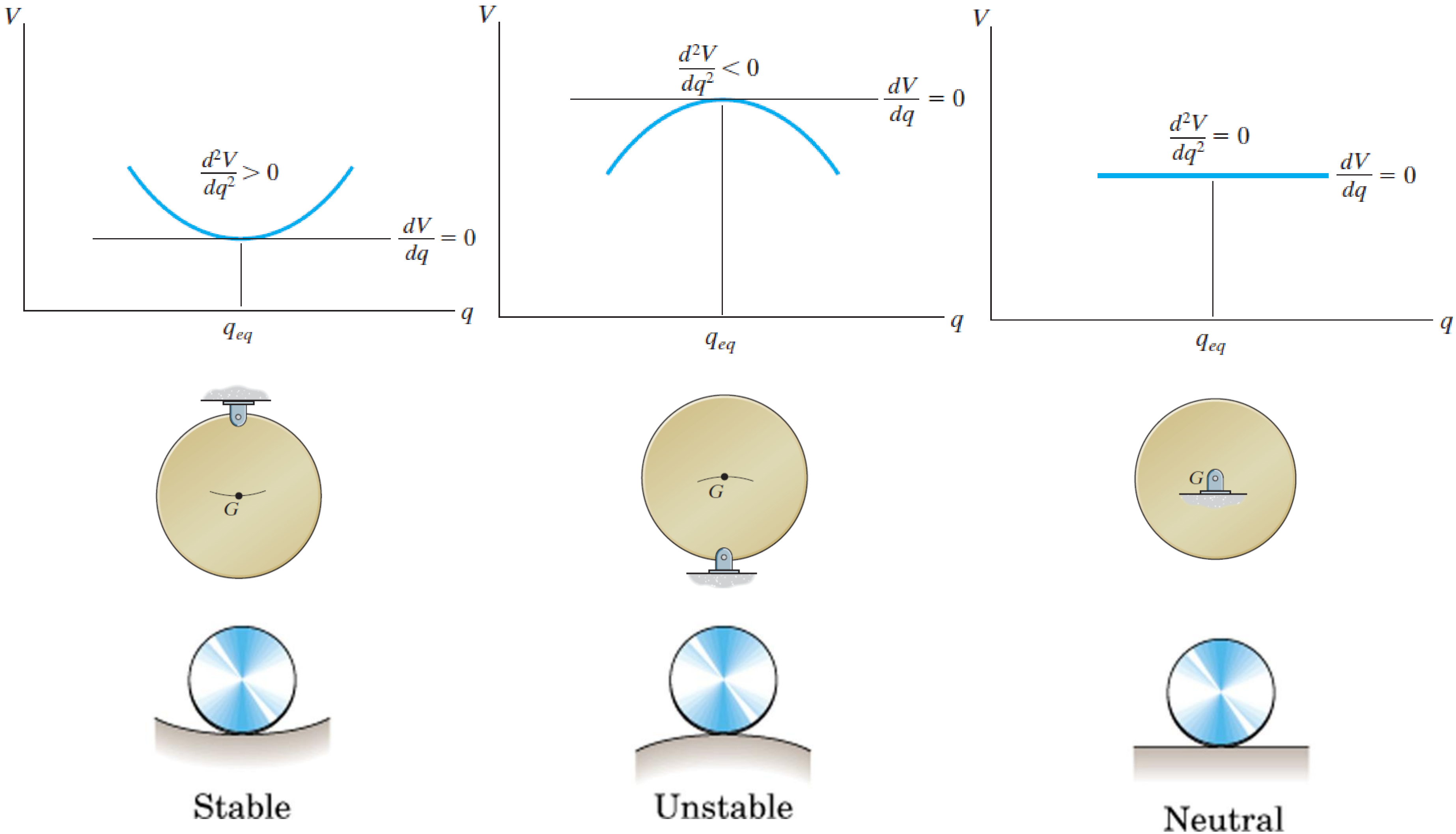
$d^2V/dx^2 = +k \rightarrow$ Stable Equilibrium since positive
Substituting values of m and k
$V = ½ (2000)x^2 - 10(9.81)x$
$x = 10(9.81)/2000 = 0.049$ m = 49 mm

Plot V vs x graph for various values of x

Minimum value of V occurs at x = 49 mm where $dV/dx = 0$ and d^2V/dx^2 is positive



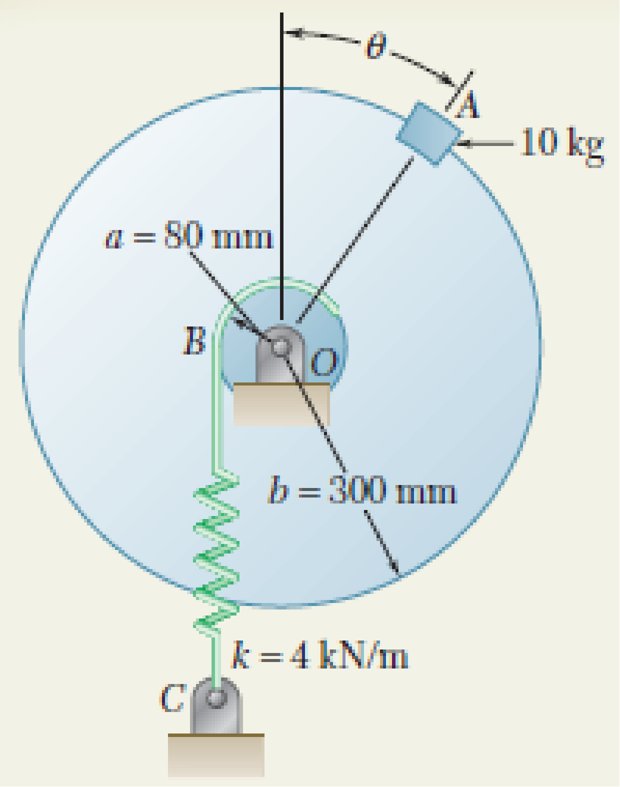
Stability of Equilibrium



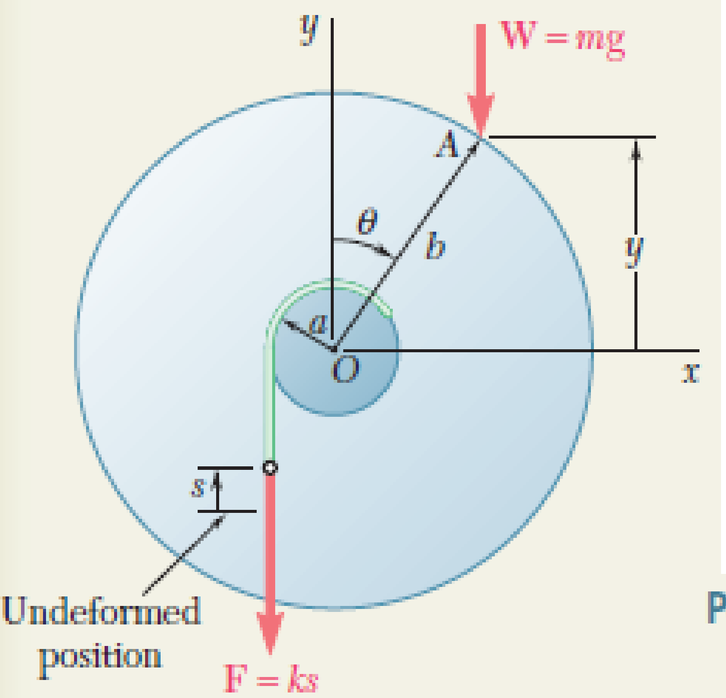
V: Minimum Maximum Constant

Example (1) on Stability of Equilibrium

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.



Example (1) on Stability of Equilibrium



$$V_e = \tfrac{1}{2}ks^2 \qquad V_g = Wy = mgy$$

Measuring θ in radians, we have

$$s = a\theta \qquad y = b\cos\theta$$

Substituting for s and y in the expressions for V_e and V_g,

$$V_e = \tfrac{1}{2}ka^2\theta^2 \qquad V_g = mgb\cos\theta$$

$$V = V_e + V_g = \tfrac{1}{2}ka^2\theta^2 + mgb\cos\theta$$

Positions of Equilibrium. Setting $dV/d\theta = 0$, we write

$$\frac{dV}{d\theta} = ka^2\theta - mgb\sin\theta = 0$$

$$\sin\theta = \frac{ka^2}{mgb}\theta$$

Substituting $a = 0.08$ m, $b = 0.3$ m, $k = 4$ kN/m, and $m = 10$ kg,

$$\sin\theta = \frac{(4\text{ kN/m})(0.08\text{ m})^2}{(10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})}\theta$$

$$\sin\theta = 0.8699\,\theta$$

Example (1) on Stability of Equilibrium

Stability of Equilibrium. The second derivative of the potential energy V with respect to θ is

$$\begin{aligned}\frac{d^2V}{d\theta^2} &= ka^2 - mgb\cos\theta \\ &= (4\ \mathrm{kN/m})(0.08\ \mathrm{m})^2 - (10\ \mathrm{kg})(9.81\ \mathrm{m/s^2})(0.3\ \mathrm{m})\cos\theta \\ &= 25.6 - 29.43\cos\theta\end{aligned}$$

For $\theta = 0$: $$\frac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 0° = -3.83 < 0$$

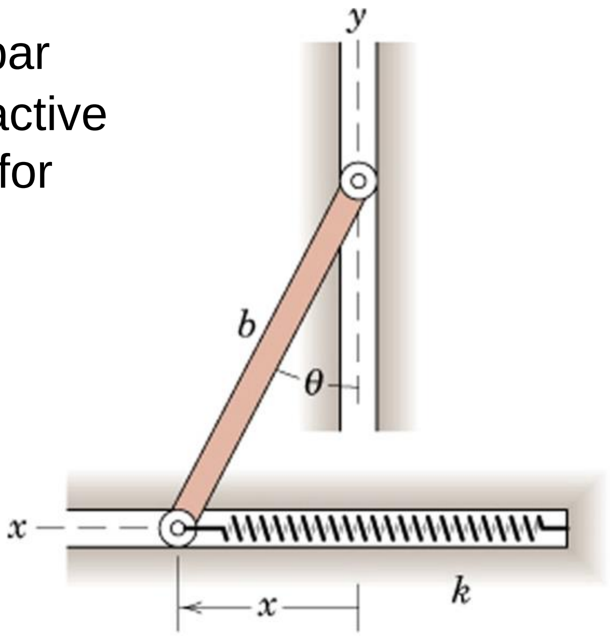
The equilibrium is unstable for $\theta = 0$ ◀

For $\theta = 51.7°$: $$\frac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 51.7° = +7.36 > 0$$

The equilibrium is stable for $\theta = 51.7°$ ◀

Example (2) on Stability of Equilibrium

Both ends of the uniform bar of mass *m* slide freely along the guides. Examine the stability conditions for the position of equilibrium. The spring of stiffness *k* is un-deformed when *x*=0



Solution: The system consists of the spring and the bar There are no external active forces. The Figure shows active force diagram. Choosing the *x*-axis as the datum plane for zero gravitational potential energy.

In the displaced position *x*:

$V_e = ½\, kx^2 = ½\, k\,(b^2 sin^2\theta)$

$V_g = mg\,(b/2)cos\theta$ (-ve of work done)

$V = V_e + V_g = ½\, k\,(b^2 sin^2\theta) + mg\,(b/2)cos\theta$

Equilibrium occurs for $\boldsymbol{dV/d\theta = 0}$

→ $kb^2 sin\theta cos\theta$ - ½ $mgbsin\theta = 0$

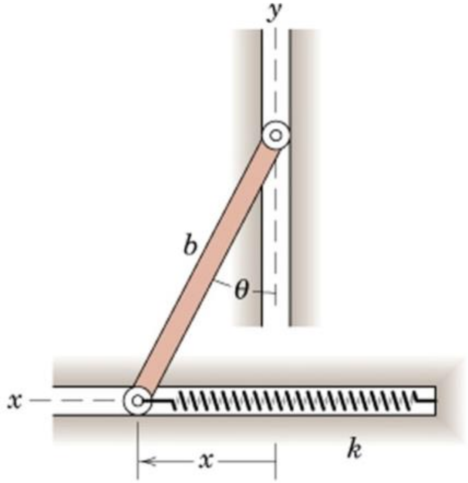
→ $(kb^2 cos\theta$ - ½ $mgb)sin\theta = 0$ → Two Solutions

→ Two equilibrium positions at:

→ $sin\theta = 0$ (and $\theta = 0$) and $cos\theta = mg/(2kb)$

Determine the stability for each of the two equilibrium position by examining the sign of $d^2V/d\theta^2$

Example (2) on Stability of Equilibrium



$d^2V/d\theta^2 = kb^2(\cos^2\theta - \sin^2\theta) - \frac{1}{2} mgb \cos\theta$
$\quad = kb^2(2\cos^2\theta - 1) - \frac{1}{2} mgb \cos\theta$

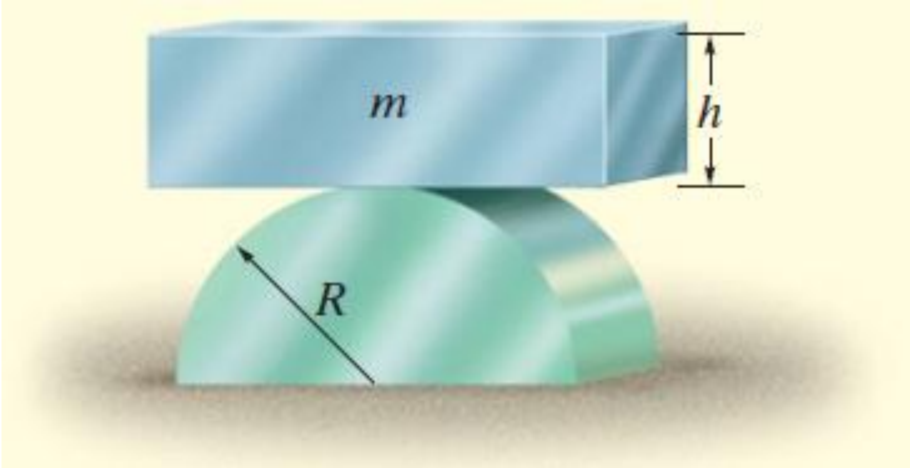
Solution I: $\sin\theta = 0$ and $\theta = 0$
$d^2V/d\theta^2 = kb^2(1 - mg/2kb)$

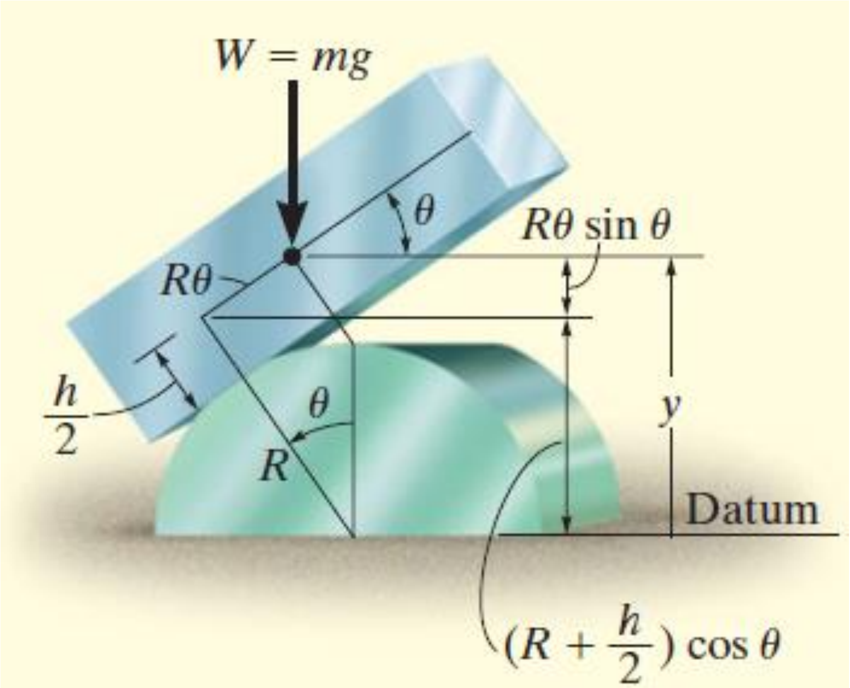
If $mg/2kb < 1$, i.e., $mg/2b < k$ → $d^2V/d\theta^2$ is positive (Stable)
If $mg/2kb > 1$, i.e., $mg/2b > k$ → $d^2V/d\theta^2$ is negative (Unstable)
→ If the spring is sufficiently stiff, the bar will return to its original vertical position, even though there is no force in the spring at that position.

Solution II: $\cos\theta = mg/(2kb)$ and $\theta = \cos^{-1}(mg/2kb)$
$d^2V/d\theta^2 = kb^2\,[(mg/2kb)^2 - 1]$
→this will be always be negative (Unstable) because $\cos\theta$, i.e., $mg/(2kb)$, must be less than unity

Virtual Work: Stability of Equilibrium

Example: A homogenous block of mass *m* rests on top surface of the cylinder. Show that this is a condition of unstable equilibrium if $h > 2R$





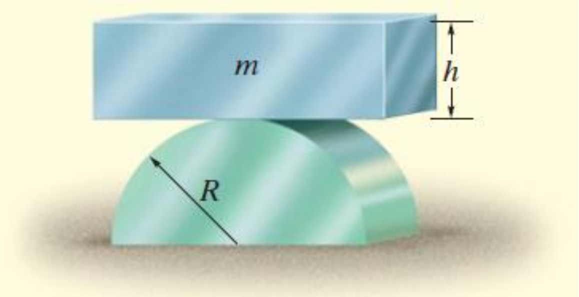
Solution: Choosing the base of the cylinder as the datum plane for zero gravitational potential energy.

$V = V_e + V_g = 0 + mgy$ (-ve of work done)

$$y = \left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta$$

Virtual Work: Stability of Equilibrium

Example: Solution

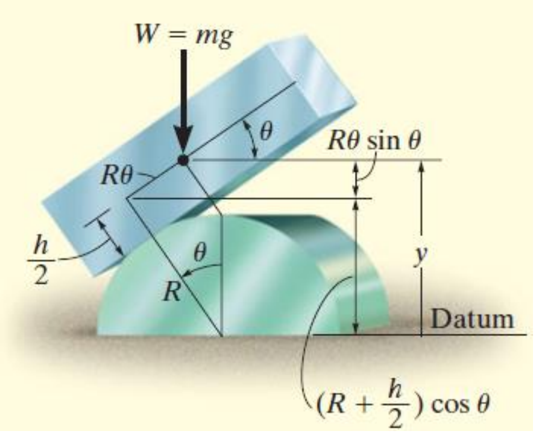


$$V = mg\left[\left(R + \frac{h}{2}\right)\cos\theta + R\theta\sin\theta\right]$$

At equilibrium:

$$\frac{dV}{d\theta} = mg\left[-\left(R + \frac{h}{2}\right)\sin\theta + R\sin\theta + R\theta\cos\theta\right] = 0$$

$$= mg\left(-\frac{h}{2}\sin\theta + R\theta\cos\theta\right) = 0$$



$\theta = 0$ is the equilibrium position that satisfies this equation

Determine the stability of the system at $\theta = 0$ by examining the sign of $d^2V/d\theta^2$

$$\frac{d^2V}{d\theta^2} = mg\left(-\frac{h}{2}\cos\theta + R\cos\theta - R\theta\sin\theta\right)$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0^\circ} = -mg\left(\frac{h}{2} - R\right)$$

Since all the constants are positive, the block is in unstable equilibrium if $h/2 > R$, i.e., $h > 2R$ because then $d^2V/d\theta^2$ will be negative.

ME101 Mid Semester Examination

Answer Sheets will be shown on 11 March 2015 during Tutorial
(Lesser number of tutorial questions that day)

•Verify your answer sheet (including total marks)

•Write “I have verified and there is no error in evaluation” on the cover page.

•Put your signature with date and return the answer sheet back to the tutor.

•Do not plead for additional marks if you do not deserve.

•In case you think that re-evaluation is necessary, submit your answer sheet to the tutor without writing the above statement and without signing.

•**Re-evaluation will be carried out and once the marks are finalized, no further request will be entertained**

•After re-evaluation, your marks may get reduced!!