SOLUTION MANUAL - TUTORIAL (I)

Q: 2) Find the determinant of matrix

$$A = \begin{bmatrix} 1 & 0 & 4 & 1 \\ -2 & 1 & -3 & 2 \\ 0 & 0 & 0 & 2 \\ 3 & 2 & 1 & -1 \end{bmatrix}$$

SOLN

$$\det(A) = \begin{vmatrix} 1 & 0 & 4 & 1 \\ -2 & 1 & -3 & 2 \\ 0 & 0 & 0 & 2 \\ 3 & 2 & 1 & -1 \end{vmatrix}$$

$$= 0(\;) - 0(\;) + 0(\;) - 2 \times \begin{vmatrix} 1 & 0 & 4 \\ -2 & 1 & -3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= (-2)\left[1(1+6) - 0 \times (\;) + 4(-4-3)\right] = \underline{\underline{42}}$$

$\boxed{10 \text{ MARKS}}$

Q: 3) Problem on Gauss-elimination method for the economy.

$$[A] = \begin{bmatrix} 0.05 & 0.09 & 0.09 & 0.19 \\ 0.16 & 0.15 & 0.28 & 0.21 \\ 0.19 & 0.21 & 0.22 & 0.27 \\ 0.27 & 0.04 & 0.35 & 0.02 \end{bmatrix}, \quad \{b\} = \begin{Bmatrix} 25 \\ 48 \\ 40 \\ 15 \end{Bmatrix} \times 10^9$$

SOLN. The Gauss elimination algorithm for an $n \times n$ matrix is:

For any $k^{th}$ step:

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik}\, a_{kj}^{(k-1)}$$

$$b_i^{(k)} = b_i^{(k-1)} - l_{ik}\, b_k^{(k-1)}$$

$$i = k+1,\ k+2,\ k+3,\ \dots,\ n$$

$$j = k,\ k+1,\ k+2,\ \dots,\ n$$

$$k = 1, 2, 3, \dots, n-1$$

$$l_{ik} = a_{ik}^{(k-1)} / a_{kk}^{(k-1)}$$

Here $n = 4$, $\therefore\ k = 1, 2, 3$

2 MARKS

Step 1 $(k=1)$

$i = 2, 3, 4$
$j = 1, 2, 3, 4$

$$l_{21} = \frac{a_{21}}{a_{11}},\quad l_{31} = \frac{a_{31}}{a_{11}},\quad l_{41} = \frac{a_{41}}{a_{11}}$$

$$a_{21}^{(1)} = 0,\quad a_{22}^{(1)} = a_{22} - l_{21} a_{12} = 0.15 - \frac{0.16}{0.05} \times 0.09 = -0.138$$

$$a_{23}^{(1)} = a_{23} - l_{21} a_{13} = 0.28 - 3.2 \times (0.09) = -0.008$$

$$a_{24}^{(1)} = a_{24} - l_{21} a_{14} = 0.21 - 3.2 \times (0.19) = -0.398$$

$$\cancel{a_{31}^{(1)} = 0,\quad a_{32}^{(1)} = a_{32} - l_{31} a}$$

$$b_2^{(1)} = b_2 - l_{21} b_1 = [48 - (3.2) \times (25)] \times 10^9 = -32 \times 10^9$$

$$a_{31}^{(1)} = 0,\quad a_{32}^{(1)} = a_{32} - l_{31} a_{12} = 0.21 - \left(\frac{0.19}{0.05}\right) \times 0.09 = -0.132$$

$$a_{33}^{(1)} = a_{33} - l_{31} a_{13} = 0.22 - 3.8 \times 0.09 = -0.122$$

$$a_{34}^{(1)} = a_{34} - l_{31} a_{14} = 0.27 - (3.8)(0.19) = -0.452$$

$$b_3^{(1)} = b_3 - l_{31} b_1 = [40 - (3.8)(25)] \times 10^9 = -55 \times 10^9$$

$$a_{41}^{(0)} = 0 , \quad a_{42}^{(1)} = a_{42} - l_{41} a_{12} = 0.04 - \left(\frac{0.27}{0.05}\right) \times 0.09$$
$$= -0.446$$

$$a_{43}^{(1)} = a_{43} - l_{41} a_{13} = 0.35 - (5.4)(0.09) = -0.136$$

$$a_{44}^{(1)} = a_{44} - l_{41} a_{14} = 0.02 - (5.4)(0.19) = -1.006$$

$$b_4^{(1)} = b_4 - l_{41} b_1 = [15 - (5.4)(25)] \times 10^9 = -120 \times 10^9$$

$\boxed{2\tfrac{1}{2} \text{ MARKS}}$

Step 2 $(k=2)$

$i = 3, 4$; $j = 2, 3, 4$

$$l_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-0.132}{-0.138} = 0.9565$$

$$l_{42} = \frac{a_{42}^{(1)}}{a_{22}^{(1)}} = \frac{-0.446}{-0.138} = 3.23188$$

$$a_{32}^{(2)} = 0 , \quad a_{33}^{(2)} = a_{33}^{(1)} - l_{32} a_{23}^{(1)} = -0.122 - (0.9565)(-0.008)$$
$$= -0.11435$$

$$a_{34}^{(2)} = a_{34}^{(1)} - l_{32} a_{24}^{(1)} = -0.452 - (0.9565)(-0.398) = -0.07131$$

$$a_{42}^{(2)} = 0 , \quad a_{43}^{(2)} = a_{43}^{(1)} - l_{42} a_{23}^{(1)} = -0.136 - (3.23188)(-0.008) = -0.11014$$

$$a_{44}^{(2)} = a_{44}^{(1)} - l_{42} a_{24}^{(1)} = -1.006 - (3.23188)(-0.398) = 0.28029$$

$$b_3^{(2)} = b_3^{(1)} - l_{32} b_2^{(1)} = [-55 - (0.9565)(-32)] \times 10^9 = -24.392 \times 10^9$$

$$b_4^{(2)} = b_4^{(1)} - l_{42} b_2^{(1)} = [-120 - (3.23188)(-32)] \times 10^9 = -16.57984 \times 10^9$$

$\boxed{1\tfrac{1}{2} \text{ MARKS}}$

Step $k = 3$

$i = 4$; $j = 3, 4$

$$l_{43} = a_{43}^{(2)} / a_{33}^{(2)} = \frac{-0.11014}{-0.11435} = 0.96318$$

$$a_{43}^{(3)} = 0 , \quad a_{44}^{(3)} = a_{44}^{(3)} - l_{43} a_{34}^{(2)} = 0.28029 - (0.96318)(-0.07131)$$
$$= 0.34897$$

$$b_4^{(3)} = b_4^{(2)} - l_{43} b_3^{(2)} = [-16.57984 - (0.96318)(-24.392)] \times 10^9$$
$$= 6.91405 \times 10^9$$

$\boxed{1 \text{ MARK}}$

Backward Substitution

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}$$

Here $n = 4, \quad i = 3, 2, 1$

$$\therefore \; x_4 = \frac{b_4^{(3)}}{a_{44}^{(3)}} = \frac{6.91405 \times 10^9}{0.34897} = \underline{\underline{19.81273 \times 10^9}}$$

$$x_3 = \frac{b_3^{(2)} - a_{34}^{(2)} \times x_4}{a_{33}^{(2)}} = \left[\frac{(-24.392) - (-0.07131) \times 19.81273}{-0.11435}\right] \times 10^9$$

$$= \underline{\underline{200.95 \times 10^9}}$$

$$x_2 = \frac{b_2^{(1)} - a_{23}^{(1)} x_3 - a_{24}^{(1)} x_4}{a_{22}^{(1)}}$$

$$= \frac{\left[-32 - (-0.008)(200.95) - (-0.398)(19.81273)\right] \times 10^9}{-0.138}$$

$$= \underline{\underline{163.0937 \times 10^9}}$$

$$x_1 = \frac{b_1 - a_{12} x_2 - a_{13} x_3 - a_{14} x_4}{a_{11}}$$

$$= \frac{\left[25 - 0.09 \times 163.0937 - 0.09 \times 200.95 - 0.19 \times 19.81273 \times 10^9\right.}{0.05}$$

$$= \underline{\underline{-230.56703 \times 10^9}}$$

$\boxed{3 \text{ MARKS}}$

Total 10 Marks.

Q.4 To use Gauss-Jordan elimination to solve

$$8x_1 + x_2 + 2x_3 - x_4 = 3$$
$$x_1 + 2x_2 - 2x_3 + x_4 = 3$$
$$-2x_1 - x_2 + 5x_3 + 3x_4 = 0$$
$$2x_1 + 3x_2 - 2x_3 - 6x_4 = -11$$

SOLN:

i.e.

$$\begin{bmatrix} 8 & 1 & 2 & -1 \\ 1 & 2 & -2 & 1 \\ -2 & -1 & 5 & 3 \\ 2 & 3 & -2 & -6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 3 \\ 0 \\ -11 \end{Bmatrix}$$

The Gauss-Jordan algorithm suggest:

At any $\underline{k^{th} \text{ step}}$:

(i) $$a_{kj}^{(k)} = \frac{a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} \quad ; \quad j = k, k+1, k+2, \ldots, n$$
$$k = 1, 2, 3, \ldots, n$$

$$a_{kk}^{(k)} = 1$$

$$b_k^{(k)} = \frac{b_k^{(k-1)}}{a_{kk}^{(k-1)}}$$

(ii) $$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k)}} = a_{ik}^{(k-1)} \quad ;$$

$$i = 1, 2, 3, \ldots, k-1, k+1, k+2, \ldots, n$$

Here $i = 1, 2, 4$ or $1, 3, 4$ or $2, 3, 4$ or $1, 2, 3$

(iii) $$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - a_{ik}^{(k-1)} a_{kj}^{(k)} \\ b_i^{(k)} &= b_i^{(k-1)} - a_{ik}^{(k-1)} b_k^{(k)} \end{aligned}\right\} \begin{aligned} i &= 1, 2, 3, \ldots k-1, k+1, \ldots n \\ j &= k, k+1, \ldots, n \end{aligned}$$

$\boxed{2 \text{ MARKS}}$

<u>Step 1</u> (i.e. $k=1$)

(i) $a_{1j}^{(1)} = \dfrac{a_{1j}}{a_{11}}$ ; $j = 1,2,3,4$

$\therefore a_{11}^{(1)} = 1, \quad a_{12}^{(1)} = \frac{1}{8} = 0.125, \quad a_{13}^{(1)} = \frac{2}{8} = 0.25,$

$a_{14}^{(1)} = \frac{-1}{8} = -0.125$

$b_1^{(1)} = \frac{3}{8} = 0.375$

(ii) $i = 2,3,4$ ; $\quad l_{21} = a_{21} = 1$

$l_{31} = a_{31} = -2$

$l_{41} = a_{41} = 2$

(iii) $\left.\begin{matrix} i = 2,3,4 \\ j = 1,2,3,4 \end{matrix}\right\}$ $\quad a_{21}^{(1)} = 0,$

$a_{22}^{(1)} = a_{22} - a_{21}a_{12} = 2 - (1 \times 0.125)$

$= \cancel{0.875}\ 1.875$

$a_{23}^{(1)} = a_{23} - a_{21}a_{13} = -2 - (1 \times 0.25) = -2.25$

$a_{24}^{(1)} = a_{24} - a_{21} \times a_{14} = 1 - (1 \times -0.125) = 1.125$

$b_2^{(1)} = b_2 - a_{21}b_1 = \cancel{-11}\ 3 - (1 \times 0.375)$

$= \cancel{-11.375}\ 2.625$

$a_{31}^{(1)} = 0,$

$a_{32}^{(1)} = a_{32} - a_{31}a_{12} = -1 - (-2 \times 0.125) = -0.750$

$a_{33}^{(1)} = a_{33} - a_{31}a_{13} = 5 - (-2 \times 0.25) = 5.500$

$a_{34}^{(1)} = a_{34} - a_{31}a_{14} = 3 - (-2 \times -0.125) = 2.750$

$b_3^{(1)} = b_3 - a_{31}b_1 = 0 - (-2 \times 0.375) = 0.750$

$a_{41}^{(1)} = 0,$

$a_{42}^{(1)} = a_{42} - a_{41}a_{12} = 3 - (2 \times 0.125) = 2.750$

$a_{43}^{(1)} = a_{43} - a_{41}a_{13} = -2 - (2 \times 0.25) = -2.500$

$a_{44}^{(1)} = a_{44} - a_{41}a_{14} = -6 - (2 \times -0.125) = -5.750$

$b_4^{(1)} = b_4 - a_{41}b_1 = -11 - (2 \times 0.375) = -11.750$

3 MARKS

Step 2 (i.e $k=2$)

(i) $a_{2j}^{(2)} = \dfrac{a_{2j}^{(1)}}{a_{22}^{(1)}}$ ; $j = 2, 3, 4$ and $a_{22}^{(1)} = 1.875$

$\therefore a_{22}^{(2)} = 1$, $a_{23}^{(2)} = \dfrac{-2.25}{1.875} = -1.20$, $a_{24}^{(2)} = \dfrac{1.125}{1.875} = 0.60$

$b_2^{(2)} = \cancel{\dfrac{11.375}{1.875}} = \cancel{-6.06667}$ $\dfrac{2.625}{1.875} = 1.40$

(ii) $i = 1, 3, 4$ ; $l_{12} = a_{12}^{(1)} = 0.125$, $l_{32} = a_{32}^{(1)} = -0.75$

$l_{42} = a_{42}^{(1)} = 2.75$

(iii) $\left.\begin{array}{l} i = 1, 3, 4 \\ j = 2, 3, 4 \end{array}\right\}$ $a_{12}^{(2)} = 0$

$a_{13}^{(2)} = a_{13}^{(1)} - l_{12}\, a_{23}^{(2)} = 0.25 - (0.125 \times -1.20) = 0.40$

$a_{14}^{(2)} = a_{14}^{(1)} - l_{12}\, a_{24}^{(2)} = -0.125 - (0.125 \times 0.60) = -0.20$

$b_1^{(2)} = b_1^{(1)} - l_{12}\, b_2^{(2)} = 0.375 - (0.125 \times \overset{1.40}{\cancel{-6.06667}}) = \overset{0.20}{\cancel{1.13333}}$

$a_{32}^{(2)} = 0$

$a_{33}^{(2)} = a_{33}^{(1)} - l_{32}\, a_{23}^{(2)} = 5.50 - (-0.75 \times -1.20) = 4.600$

$a_{34}^{(2)} = a_{34}^{(1)} - l_{32}\, a_{24}^{(2)} = 2.75 - (-0.75 \times 0.6) = 3.20$

$b_3^{(2)} = b_3^{(1)} - l_{32}\, b_2^{(2)} = 0.750 - (-0.75 \times \overset{1.40}{\cancel{-6.06667}}) = \overset{1.800}{\cancel{-3.800}}$

$a_{42}^{(2)} = 0$

$a_{43}^{(2)} = a_{43}^{(1)} - l_{42}\, a_{23}^{(2)} = -2.50 - (2.75 \times -1.20) = 0.80$

$a_{44}^{(2)} = a_{44}^{(1)} - l_{42}\, a_{24}^{(2)} = -5.75 - (2.75 \times 0.60) = -7.40$

$b_4^{(2)} = b_4^{(1)} - l_{42}\, b_2^{(2)} = -11.75 - (2.75 \times \overset{1.40}{\cancel{-6.06667}}) = \overset{-15.60}{\cancel{4.93334}}$

$\boxed{2 \text{ MARKS}}$

Step 3 (i.e. $k=3$)

(i) $a_{3j}^{(3)} = \dfrac{a_{3j}^{(2)}}{a_{33}^{(2)}}$ ; $j = 3, 4$ and $a_{33}^{(2)} = 4.60$

$\therefore a_{33}^{(3)} = 1.0$ ; $a_{34}^{(3)} = \dfrac{3.20}{4.60} = 0.69565$

$b_3^{(3)} = \dfrac{b_3^{(2)}}{4.60} = \cancel{\dfrac{-3.80}{4.60}} = \cancel{-0.82609} = \dfrac{1.80}{4.60} = 0.3913$

is :

(ii) $i = 1, 2, 4$ ; $\ell_{13} = a_{13}^{(2)} = 0.40$ , $\ell_{23} = a_{23}^{(2)} = -1.20$

$$\ell_{43} = a_{43}^{(2)} = 0.80$$

(iii) $\left.\begin{array}{l} i = 1, 2, 4 \\ j = 3, 4 \end{array}\right\}$ $a_{13}^{(3)} = 0$

$$a_{14}^{(3)} = a_{14}^{(2)} - \ell_{13} a_{34}^{(3)} = -0.20 - (0.40 \times 0.69565) = -0.47826$$

$$b_1^{(3)} = b_1^{(2)} - \ell_{13} b_3^{(3)} = \overset{0.20}{\cancel{1.13333}} - (0.40 \times \overset{0.3913}{\cancel{0.82609}}) = \cancel{0.97681} \;\; \cancel{1.46377}$$

$$= 0.04348$$

$$a_{23}^{(3)} = 0$$

$$a_{24}^{(3)} = a_{24}^{(2)} - \ell_{23} a_{34}^{(3)} = 0.60 - (-1.20 \times 0.69565) = 1.43478$$

$$b_2^{(3)} = b_2^{(2)} - \ell_{23} b_3^{(3)} = \overset{1.40}{\cancel{6.06667}} - (-1.20 \times \overset{0.3913}{\cancel{0.82609}}) = \overset{1.86956}{\cancel{7.05798}}$$

$$a_{43}^{(3)} = 0$$

$$a_{44}^{(3)} = a_{44}^{(2)} - \ell_{43} a_{34}^{(3)} = -7.40 - (0.80 \times 0.69565) = -7.95652$$

$$b_4^{(3)} = b_4^{(2)} - \ell_{43} b_3^{(3)} = \overset{-15.60}{\cancel{4.4334}} - (0.80 \times \overset{0.3913}{\cancel{0.82609}}) = \overset{-15.91304}{\cancel{5.59427}}$$

Step 4 $(k = 4)$

(i) $a_{44}^{(4)} = 1.00$ ; $b_4^{(4)} = \dfrac{b_4^{(3)}}{a_{44}^{(3)}} = \dfrac{\overset{-15.91304}{\cancel{5.59427}}}{-7.95652} = \underline{\underline{2}}$

(ii) $i = 1, 2, 3$ ; $\ell_{14} = a_{14}^{(3)} = -0.47826$ , $\ell_{24} = a_{24}^{(3)} = 1.43478$

$$\ell_{34} = a_{34}^{(3)} = 0.69565$$

(iii) $\left.\begin{array}{l} i = 1, 2, 3 \\ j = 4 \end{array}\right\}$ $a_{14}^{(4)} = 0$ ; $b_1^{(4)} = b_1^{(3)} - \ell_{14} b_4^{(4)} = \overset{0.04348}{\cancel{1.46377}} - (-0.47826 \times 2)$

$$= \underline{1}$$

$$a_{24}^{(4)} = 0, \quad b_2^{(4)} = b_2^{(3)} - \ell_{24} b_4^{(4)} = \overset{1.86956}{\cancel{7.05798}} - (1.43478 \times 2)$$

$$= \underline{\underline{-1}}$$

$$a_{34}^{(4)} = 0, \quad b_3^{(4)} = b_3^{(3)} - \ell_{34} b_4^{(4)} = \overset{0.3913}{\cancel{0.82609}} - (0.69565 \times 2)$$

$$= \underline{\underline{-1}}$$

Solution Vector : $\left\{ \begin{array}{c} 1 \\ -1 \\ -1 \\ 2 \end{array} \right\}$

$\boxed{3 \text{ MARKS}}$