Lecture 9: ITERATIVE METHODS (13-Aug-2012)

ITERATIVE METHODS

→ As discussed earlier many large systems of linear algebraic equations have extremely sparse $[A]$.

→ They may be diagonally dominant

Q: What is diagonally dominant matrix?

If in

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$|a_{ii}| \geq \sum_{j=1,\, j\neq i}^{n} |a_{ij}|,$$ then the matrix is diagonally dominant.

→ Diagonally dominant matrices can be solved by iterative methods.

→ Jacobi iteration
→ Gauss Siedel iteration
→ Successive Over Relaxation

* The methodology in any iterative scheme is
  → Assume an initial solution vector $\{x\}^{(0)}$
  → Using this initial $\{x\}^{(0)}$ values improved solution vector $\{x\}^{(1)}$ is obtained.
  → Again using $\{x\}^{(1)}$ improved $\{x\}^{(2)}$ is obtained.
  → The process goes on till $\{x\}^{(s)}$ converges to actual solution.

* Please note that diagonal dominance is essential.

Jacobi Iteration

$[A]\{x\} = b$ given linear system.

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad ; \quad i = 1, 2, 3 \ldots, n$$

e.g. First row is

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1j} x_j + \cdots + a_{1n} x_n = b_1$$

Ily i^{th} row

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{ii} x_i + \cdots + a_{in} x_n = b_i$$

Here $$x_i = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j \right);$$

$$i = 1, 2, 3, \ldots, n$$

An initial solution vector is proposed $\{x\}^{(0)}$

→ The iterative scheme works in such a way that

$$x_i^{(1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(0)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(0)} \right);$$

$i = 1, 2, 3, \ldots, n$

→ The procedure is repeated.

For any $(s+1)^{th}$ iteration, the Jacobi iteration scheme is

$$x_i^{(s+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(s)} \right);$$

$i = 1, 2, 3, \ldots, n.$

This can be given as:

$$x_i^{(s+1)} = x_i^{(s)} + \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(s)} - a_{ii} x_i^{(s)} \right)$$

$$= x_i^{(s)} + \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^{(s)} \right)$$

$(i = 1, 2, 3, \ldots, n)$

Theoretically speaking if x_i is the actual solution, then

$$b_i - \sum_{j=1}^{n} a_{ij} x_j = 0$$

However, as we are using iterative scheme the solution at any iteration $x_i^{(s)}$ may not be accurate.

We define a residual quantity

$$R_i^{(s)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(s)}$$

$\therefore$ In Jacobi iterative scheme.

$$x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{i,i}}$$

$$i = 1, 2, 3, \ldots, n$$
$$s = 1, 2, 3, \ldots$$

$R_i^{(s)} \rightarrow$ Residual of equation i.

$\rightarrow$ In Jacobi method all values are simultaneously iterated.

Example

Demonstrate few iterative steps using Jacobi iteration scheme to solve the following linear system

$$\begin{bmatrix} 5 & 0 & -2 \\ 3 & 5 & 1 \\ 0 & -3 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 2 \\ -4 \end{Bmatrix}$$

Solution

As per Jacobi's iterative scheme, the i^{th} solution of the solution vector $\{x\}$ is given as:

$$x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{ii}}$$

We have here $x_i \rightarrow x_1, x_2, x_3$

and $R_i^{(s)} = b_i - \sum_{j=1}^{3} a_{ij} x_j^{(s)}$

Let us initially assume $\{x\}^{(0)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ i.e. $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$

s	$R_i^{(s-1)}$	$x_i^{(s)}$
1	$R_1^{(0)} = b_1 - \left(a_{11} x_1^{(0)} + a_{12} x_2^{(0)} + a_{13} x_3^{(0)}\right)$ $= 7 - \left[5\times 0 + 0\times 0 + (-2\times 0)\right] = \underline{\underline{7}}$	$x_1^{(1)} = 0 + \frac{1}{5} \times 7 = 1.40$
	$R_2^{(0)} = b_2 - \left(a_{21} x_1^{(0)} + a_{22} x_2^{(0)} + a_{23} x_3^{(0)}\right)$ $= 2 - \left[3\times 0 + 5\times 0 + 1\times 0\right] = \underline{\underline{2}}$	$x_2^{(1)} = 0 + \frac{1}{5} \times 2 = 0.40$
	$R_3^{(0)} = b_3 - \left(a_{31} x_1^{(0)} + a_{32} x_2^{(0)} + a_{33} x_3^{(0)}\right)$ $= -4 - \left[0 + (-3\times 0) + (4\times 0)\right] = \underline{\underline{-4}}$	$x_3^{(1)} = 0 + \frac{1}{4} \times (-4) = -1.00$
2	$R_1^{(1)} = 7 - \left[5\times 1.4 + 0 + (-2\times -1)\right] = -2$	$x_1^{(2)} = 1.40 + \frac{(-2)}{5} = 1.00$
	$R_2^{(1)} = 2 - \left[3\times 1.4 + 5\times 0.4 + (1\times -1)\right] = -3.2$	$x_2^{(2)} = 0.40 + \frac{(-3.2)}{5} = -0.24$
	$R_3^{(1)} = -4 - \left[0 + (-3\times 0.4) + (4\times -1)\right] = 1.2$	$x_3^{(2)} = -1.0 + \frac{1.2}{4} = -0.70$
3	$R_1^{(2)} = 7 - \left[5\times 1.0 + 0 + (-2\times -0.70)\right] = 0.60$	$x_1^{(3)} = 1.0 + \frac{0.60}{5} = 1.12$
	$R_2^{(2)} = 2 - \left[3\times 1.0 + 5\times(-0.24) + (1\times -0.70)\right]$ $= 0.90$	$x_2^{(3)} = -0.24 + \frac{0.9}{5} = -0.06$
	$R_3^{(2)} = -4 - \left[0 + (-3\times -0.24) + (4\times -0.70)\right]$ $= -1.92$	$x_3^{(3)} = -0.70 + \frac{(-1.92)}{4} = -1.18$

The iterations have to continued by the same process till the solution converges.

Accuracy, Convergence, etc. of Iterative Methods

→ As told earlier non-singular system of linear algebraic equations have exact solutions

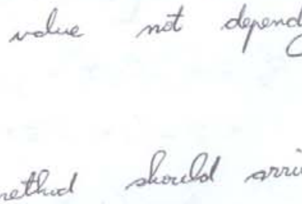
→ In direct elimination methods, you tried to find these exact solutions (and it is more or less attainable).

→ However, the problem of round-off errors were there that could disrupt the solution.

→ One process to overcome was pivoting.

→ The iterative methods are less susceptible to round-off errors
(∵ Each iteration starts with a value not depending on the round-offs.)

→ Theoretically speaking, your iterative method should arrive at the exact solution asymptotically as the no. of iteration increases.



We will try to maintain accuracy based on the requirement of scheme.
(Machine accuracy not important highlighted).
Accuracy is measured in terms of the error of the method.

Absolute Error = Approximated value - Exact Value

Relative Error = $\frac{\text{Absolute Error}}{\text{Exact Value}}$

Convergence is achieved when the desired accuracy is achieved.

We may not know the exact soln. in iterative methods.

$\therefore$ Error at any ~~step~~ iterative step $= x_i^{(s+1)} - x_i^{(s)}$

$= \Delta x_i$

→ If the solutions are exact, residuals should be zero.

i.e. $x_i^{(s+1)} - x_i^{(s)} = \frac{R_i^{(s)}}{a_{ii}}$

$$|(\Delta x_i)_{max}| = \frac{R_i^{(s)}}{a_{ii}}$$

$$|(\Delta x_i)_{max}| \leq \varepsilon \quad \text{or} \quad \sum_{i=1}^{n} |\Delta x_i| \leq \varepsilon$$

$$\text{or} \quad \left[\sum_{i=1}^{n} (\Delta x_i)^2\right]^{1/2} \leq \varepsilon$$

In relative case: $\left|\frac{(\Delta x_i)_{max}}{x_i}\right| \leq \varepsilon$; $\sum_{i=1}^{n} \left|\frac{\Delta x_i}{x_i}\right| \leq \varepsilon$;

$$\left[\sum_{i=1}^{n} \left(\frac{\Delta x_i}{x_i}\right)^2\right]^{1/2} \leq \varepsilon .$$

Gauss- Siedel Iteration

In Jacobi method

$$x_i^{(s+1)} = x_i^{(s)} + \frac{1}{a_{ii}}\left[b_i - \sum_{i=1}^{n} a_{ij} x_j^{(s)} \right]$$

If you are sequentially finding solution from $i = 1, 2, 3, \dots, n$

then at i^{th} case:

You already know values of $x_1^{(s+1)}, x_2^{(s+1)}, \dots, x_{i-1}^{(s+1)}$.

→ Utilising this:

$$x_i^{(s+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(s)} \right];$$

$i = 1, 2, 3, \dots, n.$

→ This is method of successive iteration.

You can also say:

$$x_i^{(s+1)} = x_i^{(s)} + \frac{R_i^{(s)}}{a_{ii}} \quad ; \quad i = 1, 2, 3, \dots, n$$

$$R_i^{(s)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(s+1)} - \sum_{j=i}^{n} a_{ij} x_j^{(s)} \quad ; \quad i = 1, 2, 3, \dots, n$$