# Lecture 8: Drawbacks in Elimination Method

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LECTURE 8
08 - AUGUST - 2012

Last class we discussed on

→ Banded Matrix and Sparse Matrix
→ Thomas Algorithm

## Drawbacks in Elimination Methods

Main drawbacks are:

→ Cumulate Round-off errors
→ Not suitable for ill-conditioned systems.

### Round-off Errors

* Computers can store numbers with finite precision only (say single precision with 7 significant digits, double precision with 14 significant digits, etc).

* If you have infinite precision → then round-off errors will not appear.

* Due to approximating infinite precision numbers using finite-precision, we get round-off errors.

Consider the linear algebraic system

$$0.0003\, x_1 + 3\, x_2 = 1.0002$$
$$x_1 + x_2 = 1$$

Using traditional Gauss elimination,

$$\left[\begin{array}{cc|c} 0.0003 & 3 & 1.0002 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - \left(\frac{1}{0.0003}\right)R_1} \Rightarrow \left[\begin{array}{cc|c} 0.0003 & 3 & 1.0002 \\ 0 & -9999 & -3333 \end{array}\right]$$

The actual solutions should be

$x_2 = 0.333333 \ldots\ldots$, and $x_1 = 0.666666 \ldots$

However in computer you have to restrict to finite precision.

You know $x_2 = \dfrac{1 - \frac{1.0002}{0.0003}}{9999}$ and $x_1 = \dfrac{1.0002 - 3x_2}{0.0003}$

Let us compare the results if you are ~~using~~ varying significant figures from 4 to 7.

| Significant Fig | $x_2$ | $x_1$ |
|---|---|---|
| 4 | 0.3333 | 1.0000 |
| 5 | 0.33333 | 0.7000 |
| 6 | 0.333333 | 0.6700 |
| 7 | 0.3333333 | 0.6670 |

∴ Based on your precision, you can see the variation in solution. This can be overcomed by partial pivoting.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0.0003 & 3 & 1.0002 \end{array}\right] \xrightarrow{R_2 = R_2 - 0.0003R_1} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2.9997 & 0.9999 \end{array}\right]$$

| Sign Fig | $x_2$ | $x_1$ |
|---|---|---|
| 4 | 0.3333 | 0.6667 |
| 5 | 0.33333 | 0.66667 |
| 6 | 0.333333 | 0.666667 |
| 7 | 0.3333333 | 0.6666667 |

$(x_1 = 1 - x_2)$

System Condition

→ Usually well-posed non-singular systems have exact solutions.

→ As in computers you can store only finite precision numbers, chances of round-off errors are more.

→ Systems that are sensitive to small change in numbers due to round-off errors may give erroneous solutions by elimination methods. They are not suitable.

→ A well-conditioned problem is one in which a small change in any of the elements causes only small change in the solution of the problem.

→ An ill-conditioned problem is the one where small changes in numerical values causes large change in solution. (Usually ill-conditioned systems are sensitive to round-off errors).

See the example as referred from Hoffman (2001) below:

$$x_1 + x_2 = 2$$
$$x_1 + 1.0001\,x_2 = 2.0001$$

i.e. $\left[\begin{array}{cc:c} 1 & 1 & 2 \\ 1 & 1.0001 & 2.0001 \end{array}\right] R_2 = R_2 - \left(\frac{1}{1}\right)R_1 \Rightarrow \left[\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & 0.0001 & 0.0001 \end{array}\right]$

$\therefore \boxed{x_2 = 1 \quad \text{and} \quad x_1 = 1}$

If we slightly modify the system as:

$$x_1 + x_2 = 2$$
$$x_1 + 0.9999\,x_2 = 2.0001$$

i.e. $\left[\begin{array}{cc:c} 1 & 1 & 2 \\ 1 & 0.9999 & 2.0001 \end{array}\right] R_2 = R_2 - \left(\frac{1}{1}\right)R_1 \Rightarrow \left[\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & -0.0001 & 0.0001 \end{array}\right]$

$\therefore \boxed{x_2 = -1, \quad \text{and} \quad x_1 = 3}$

There is quite a significant change in the solution of the given system, eventhough only slight variation is made in the system. This is because the system is ill-conditioned.

* With infinite precision, it may be possible to overcome ill-condition. However, in computational solutions you need to take care of avoiding ill-conditioned systems.

* Ill-conditioning can be determined by Condition Number.

Norms and Condition Number

Norms : → It is the measure of magnitude of a matrix or vector like [A], {x}, {b}, etc.

(I am avoiding [ ], { } inside the norm symbols)

$\|A\|$, $\|x\|$, $\|b\|$ - etc..

Please note that

$$\|A\| > 0 \quad \rightarrow (i)$$

If $\|A\| = 0$ then it is due to $[A] = 0$ → (ii)

A scalar quantity $k$ if it is multiplied to matrix A we get $[kA]$

$$\|kA\| = |k| \, \|A\| \quad \rightarrow (iii)$$

↳ Mode or Absolute

[A], [B] exist

$$\|A + B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\|\,\|B\|$$

Norm of a scalar is its absolute value $\|k\| = |k|$

Norm of vector $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = x_i$

$$\|x\|_1 = \Sigma |x_i| \quad \text{(Sum of the magnitudes)}$$

$$\|x\|_2 = \|x\|_e = \left(\Sigma x_i^2\right)^{1/2} \quad \text{(Euclidean Norm)}$$

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad \text{Maximum magnitude norm.}$$

For an $n \times n$ matrix $[A]$

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad \rightarrow \text{Maximum of the column sums.}$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}| \quad \rightarrow \text{Max. of row sums.}$$

$$\|A\|_e \rightarrow \text{Euclidean Norm} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

Condition Number

→ It is the measure of sensitivity of the system to small changes in any of its elements.

Consider the linear system $[A]\{x\} = \{b\}$

$$\therefore \|b\| \leq \|A\|\,\|x\| \quad \text{(Isn't it?)}$$

Let us change the R.H.S vector $\{b\}$ by a small amount
$\{b\} + \{\delta b\}$.

This will cause change in solution $\{x\} + \{\delta x\}$

$$\therefore \quad [A](\{x\} + \{\delta x\}) = \{b\} + \{\delta b\}$$

or $$[A]\{x\} + [A]\{\delta x\} = \{b\} + \{\delta b\}$$

or $$[A]\{\delta x\} = \{\delta b\}$$

or $$\{\delta x\} = [A]^{-1}\{\delta b\}$$

$$\|\delta x\| \leq \|A^{-1}\| \, \|\delta b\|$$

or Multiplying on L.H.S by $\|b\|$ and RHS by $\|A\| \, \|x\|$

$$\therefore \quad \|b\| \, \|\delta x\| \leq \|A\| \, \|x\| \, \|A^{-1}\| \, \|\delta b\|$$

i.e. $$\frac{\|\delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}$$

i.e. $$\frac{\|\delta x\|}{\|x\|} \leq C(A) \frac{\|\delta b\|}{\|b\|}$$

where $C(A) = \|A\| \, \|A^{-1}\|$ is called the condition number.

Smaller values of $C(A)$ → small sensitivity of system
(well conditioned)

Larger values of $C(A)$ → large sensitivity (ill-conditioned).

Q: Check the following matrix $[A] = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$

~~See its~~ Find its conditioning.

$$[A] = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$$

$$\therefore [A]^{-1} = 10000 \begin{bmatrix} 1.0001 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10001 & -10000 \\ -10000 & 10000 \end{bmatrix}$$

$$\|A\|_e = 2.00005 \quad , \quad \|A^{-1}\| = 20000.50002$$

$$\therefore \quad C(A) = \|A\| \, \|A^{-1}\| = \underline{\underline{40002}} \quad \ggg 1$$

Ill-conditioned.

Quiz-2

1) What is meant by bandwidth of a multi-diagonal sparse matrix. What is the bandwidth of a tri-diagonal matrix?

2) Which is the element that gets modified during elimination of a tri-diagonal matrix?

3) What is condition number?