Lecture 7: Banded Matrices and Thomas Algorithm

CE 601 NUMERICAL METHODS

LECTURE 7

07-AUGUST-2012

In the last class we discussed on

→ LU Decomposition

→ The algorithm for Doolittle's method

There are further topics on LU Decomposition.
I request you to refer the books to understand

→ Crout's method

→ Cholesky's method for symmetric and positive definite coefficient matrix.

* Today we will discuss on solving systems, where the matrices have certain properties

Banded Matrices

→ You have seen that the coefficient matrix [A] consist of $n \times n$ elements.

→ Mostly we have dealt with (till now) dense matrices. (i.e. Majority of elements in the matrix are non-zeros).

* However, in reality when dealing with many engineering problems, you may come across coefficient matrices having many zero elements. Such matrices are called Sparse Matrix

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\rightarrow \text{ Dense Matrix}} \quad \text{Most of } a_{ij} \neq 0 \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \cdots & a_{1n} \\ 0 & a_{22} & 0 & \cdots & a_{2n} \\ 0 & & \vdots & & 0 \\ 0 & & & & \vdots \\ \vdots & & & & \\ a_{n-1,1} & & & & \\ a_{n1} & & - & - - & a_{nn} \end{bmatrix}}_{\downarrow \text{ A Sparse Matrix}}$$

* In Gauss elimination method you have seen that you need to perform $(n-1)$ row operations.

→ If there are many zeros in $[A]$, then there is no point in conducting all these row operations.

→ Moreover zero is occupying the memory space allotted for the ~~new~~ matrix.

~~Say if~~

* ~~A~~ Special easier methods are available to solve sparse matrices that are Banded.

* The non-zero elements are along the main diagonal as well as lines parallel to main diagonal

$$\begin{array}{cc} & \overset{\longleftarrow\ w_c\ \longrightarrow}{} \\ w_r & \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 & 0 & \cdots & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & 0 & \cdots & 0 \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & 0 \cdots & 0 \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68}\ 0 \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ \ddots & & & & & & & & a_{nn} \end{bmatrix} \end{array}$$

* Bandwidth $w = w_c + w_r - 1$.

$$a_{ij} = \begin{cases} 0 & ; \ j \geq i + w_c \\ 0 & ; \ i \geq j + w_r \\ a_{ij} & \text{else} \end{cases}$$

* A special type of banded matrix when

$w_c = 2$, $w_r = 2$, $\therefore\ w = 2+2-1 = 3$

This is tri-diagonal matrix.

* A system having tri-diagonal matrix is

$$[T]\{x\} = \{b\}$$

$$
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & 0 & \cdots & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & \cdots & 0 \\
0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & 0 \\
0 & 0 & a_{43} & a_{44} & a_{45} \; 0 & \cdots & 0 \\
\vdots & & & & & & \\
\vdots & & & & & & \\
0 & \cdots & & & & a_{n,n-1} & a_{nn}
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{Bmatrix}
=
\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{Bmatrix}
$$

Thomas Algorithm

Thomas (1949) formulated a method to solve tri-diagonal system.

Recall Gauss elimination method

If you apply Gauss elimination for tri-diagonal matrix

$$
\begin{bmatrix}
a_{11} & a_{12} & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 \\
0 & a_{32} & a_{33} & a_{34} & 0 \\
0 & 0 & a_{43} & a_{44} & a_{45} \\
0 & 0 & 0 & a_{54} & a_{55}
\end{bmatrix}
\qquad
\begin{aligned}
R_2 &= R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1 \\
&\cancel{R_3 = R_3} \; \cdots
\end{aligned}
$$

→ Only the second row requires row operation in $k = 1$ step.

→ Due to this operation the IInd row becomes

$$
\left[\; 0 \quad a_{22} - \left(\frac{a_{21}}{a_{11}}\right) a_{12} \quad a_{23} \quad 0 \quad 0 \;\right]
$$

→ That is only the diagonal element of that row is getting changed.

i.e. For tri diagonal matrix

* At each step, k

→ ~~the rows~~ Only the $(k+1)^{th}$ row is modified

→ And only the diagonal element $a_{k+1,k+1}$ is modified.

i.e. $\cancel{a_{k+1,k+1} = a_{k+1,k+1}}$

$$a^{(k)}_{i,i} = a^{(k-1)}_{i,i} - \left(\frac{a^{(k-1)}_{i,j-1}}{a^{(k-1)}_{i-1,i-1}} \right) a^{(k-1)}_{i-1,j}$$

where
$$\left. \begin{aligned} i &= k+1 \\ j &= k+1 \end{aligned} \right\} \rightarrow \text{only.}$$

* R.H.S. vector b

$$b^{(k)}_{i} = b^{(k-1)}_{i} - \left(\frac{a^{(k-1)}_{i,j-1}}{a^{(k-1)}_{i-1,i-1}} \right) b^{(k-1)}_{i-1}$$

So keeping this in mind, now we can take each equation of tri diagonal system.

$$\left. \begin{aligned} a_{11} x_1 + a_{12} x_2 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{32} x_2 + a_{33} x_3 + a_{34} x_4 &= b_3 \end{aligned} \right\}$$

$n \times n$ matrix can be stored as $n \times 3$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix} \Rightarrow \begin{bmatrix} - & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} \\ a_{43} & a_{44} & a_{45} \\ a_{54} & a_{55} & a_{56} \\ a_{65} & a_{66} & - \end{bmatrix}$$

Let me call this as matrix A'

$$A' = \begin{bmatrix} - & a'_{12} & a'_{1,3} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \\ a'_{41} & a'_{42} & a'_{43} \\ a'_{51} & a'_{52} & a'_{53} \\ a'_{61} & a'_{62} & \end{bmatrix}$$

where $a'_{12} = a_{11}$, $a'_{22} = a_{22}$, $a'_{32} = a_{33}$, $a'_{42} = a_{44}$, etc.

$\Rightarrow$ The computations can be proceeded as such:

$$a'_{1,2} = a'_{1,2}$$

Recall the portion in previous page.

$$a'_{i,2} = a'_{i,2} - \left(\frac{a'_{i,1}}{a'_{i-1,2}} \right) a_{i-1,3}$$

for $i = 2, 3, 4, 5, \ldots, n$

$\Rightarrow$ Vector $\{b\}$ is modified as:

$$b_1 = b_1$$

$$b_i = b_i - \left(\frac{a'_{i,1}}{a'_{i-1,2}} \right) b_{i-1} \quad ; \quad i = 2, 3, 4, \ldots, n$$

$\Rightarrow$ The multiplying factor $\left(\dfrac{a'_{i,1}}{a'_{i-1,2}} \right)$ can be stored.

$\Rightarrow$ Using back-substitution:

$$x_n = b_n / a'_{n,2}$$

$$x_i = \frac{b_i - a'_{i,3}\, x_{i+1}}{a'_{i,2}}$$

$(i = n-1, n-2, \ldots, 1)$

Example:

Solve the tri-diagonal system using Thomas Algorithm.

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 1 \\ 2 \\ -2 \end{Bmatrix}$$

Given $[A] = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Now $[A'] = \begin{bmatrix} - & -2 & 1 \\ 1 & -4 & 1 \\ 1 & -4 & 1 \\ 1 & -2 & - \end{bmatrix}$
(Not Transpose)

As only diagonal elements get modified. $a'_{i,2} = a'_{i,2} - \left(\frac{a'_{i,1}}{a'_{i-1,2}}\right) a_{i-1,3}$

for $i = 2, 3, 4$

$a'_{1,2} = -2$; $b_1 = 3$

$a'_{2,2} = -4 - \left(\frac{1}{-2}\right) \times 1 = -3.5$; $b_2 = 1 - \left(\frac{1}{-2}\right) \times 3 = 2.5$

$a'_{3,2} = -4 - \left(\frac{1}{-3.5}\right) \times 1 = -3.71429$ $\left(= a'_{32} - \left(\frac{a'_{3,1}}{a'_{2,2}}\right) a_{2,3} \right)$;

$b_3 = 2 - \left(\frac{1}{-3.5}\right) \times 2.5 = 2.71429$

$a'_{4,2} = -2 - \left(\frac{1}{-3.71429}\right) \times 1 = -1.73077$;

$b_4 = -2 - \left(\frac{1}{-3.71429}\right) \times 2.71429$

$= -1.26923$

$$\therefore \quad x_4 = \frac{b_4}{a'_{4,2}} = \frac{-1.26923}{-1.73077}$$

$$= \underline{\underline{0.73333}}$$

$$x_3 = \frac{b_3 - a'_{3,3}\,x_4}{a'_{3,2}} = \frac{2.71429 - 1 \times 0.73333}{(-3.71429)}$$

$$= \cancel{1.98096} \; -0.53333$$

$$x_2 = \frac{b_2 - a'_{2,3}\,x_3}{a'_{2,2}} = \frac{2.5 - 1 \times \cancel{1.98096}\,(-0.53333)}{(-3.5)}$$

$$= -0.86667$$

$$x_1 = \frac{b_1 - a'_{1,3}\,x_2}{a'_{1,2}} = \frac{3 - 1 \times (-0.86667)}{-2}$$

$$= \underline{\underline{-1.93333}}$$