Lecture 6: LU Decomposition

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In the last class we discussed on:

→ Gauss-Jordan elimination method

→ We worked out an example problem as well.

LU Decomposition

* We have earlier discussed that matrices can be factored.

$$[A] = [B]\,[C] \longrightarrow (1)$$

* There can be infinite number of combinations possible for the above equation (1).
* One can easily then decompose or factorise [A] into a lower triangle matrix and upper triangle matrix.

i.e. $$[A] = [L]\,[U] \longrightarrow (2)$$

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 \cdots & 0 \\ \vdots & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Q: What is the purpose of decomposing the matrix [A] ?

→ If we need to solve several linear systems with same coefficient matrix, then it will be tedious to to do Gauss elimination for every R.H.S. vector.

→ Especially in a situation where the R.H.S vector is/are a functions of the previous solution vectors. ~~of previous~~

→ To counter such unnecessary computations LU Decomposition is recommended.

<u>Please note about LU Decomposition</u>

$$[A]\{x\} = \{b\} \rightarrow (1)$$

i.e. $$[L][U]\{x\} = \{b\} \rightarrow (2)$$

Now multiply by $[L]^{-1}$;

$$[L]^{-1}[L][U]\{x\} = [L]^{-1}\{b\}$$

i.e. $$[I][U]\{x\} = [L]^{-1}\{b\}$$

or $$[U]\{x\} = [L]^{-1}\{b\} \rightarrow (3)$$

Let us define $[U]\{x\} = \{c\} \rightarrow (4)$

$\therefore$ We get $\{c\} = [L]^{-1}\{b\}$

or $[L]\{c\} = \{b\} \rightarrow (5)$

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* That is, the lower triangle matrix $[L]$ will transform the R.H.S. vector $\{b\}$ to $\{c\}$ using the relation $\{c\} = [L]^{-1}\{b\}$
* After obtaining $\{c\}$, using the relation $[U]\{x\} = \{c\}$, we get the solution vector.

LU Decomposition Forms

$$A_0 \quad [A] = [L][U]_n \rightarrow (1)$$

there are many combinations of this eqn. (1).

$\Rightarrow$ Now if the diagonal elements of $[L]$ are 1, then the combination $[A] = [L][U]$ will be unique.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & & \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{bmatrix}$$

The ~~formation~~ solution of linear system of Eq unique combination is Doolittle's method.

$\Rightarrow$ If the diagonal elements of $[U]$ are 1, again the decomposition $[A] = [L][U]$ will be unique

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

This method of solving the system is called Crout's method.

Quick Example (Hoffman, 2001)

We will discuss the method of decomposition later. Let us do a quick example.

$$\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}$$ Solve this system by Doolittle method.

$$[A]\{x\} = \{b\}$$

Now $[A] = [L]\,[U]$

$$\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix}$$

(Check the product yourself) → The method for decomposition to be explained later.

Now $[L]\{c\} = \{b\}$

i.e. $$\begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -5/7 & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \\ 20 \end{Bmatrix}$$

Use forward substitution,

$$c_1 = 20$$

$$c_2 = 20 - \left(\frac{-20}{4}\right) = 25$$

$$c_3 = 20 - \left(\frac{-20}{4}\right) - \left(\frac{-5}{7} \times 25\right) = \frac{300}{7}$$

Now $[U]\{x\} = \{c\}$

i.e. $$\cancel{\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 25 \\ 300/7 \end{Bmatrix}} \quad \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 25 \\ 300/7 \end{Bmatrix}$$

$$\therefore \; x_3 = 0.40, \quad x_2 = \frac{25 + 25 \times 0.4}{35} = 1.0$$

$$x_1 = 0.60$$

Doolittle Method for LU Decomposition

Recall your Gauss elimination algorithm:

→ At any k^{th} step:

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik}\, a_{kj}^{(k-1)}$$

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$$

$$\left.\begin{array}{l} i = k+1, k+2, \dots, n \\ j = k, k+1, k+2, \dots, n \\ k = 1, 2, 3, \dots, n-1 \end{array}\right.$$

→ Doing this elimination, you are finally getting an upper triangular matrix using Gauss elimination.

$$[A] \longrightarrow [U] \quad \text{in } (n-1) \text{ steps.}$$

Now you can see that

$$a_{ij}^{(k)} - a_{ij}^{(k-1)} = -\, l_{ik}\, a_{kj}^{(k-1)}$$

(i.e. Elements in matrix A gets modified).

As you know $k = 1, 2, 3, 4, \dots, (n-1)$

$$a_{ij}^{(k)} - a_{ij}^{(k-2)} = -\, l_{ik}\, a_{kj}^{(k-1)} - l_{i,k-1}\, a_{k-1,j}^{(k-2)} \quad \text{so on}$$

$|||^{ly}$ $$a_{ij}^{(k)} - a_{ij} = -\sum_{k=1}^{k} l_{i,k}\, a_{k,j}^{(k-1)} \; ; \quad \begin{array}{l} i = k+1, k+2, \dots, n \\ j = k, k+1, k+2, \dots, n \end{array}$$

$$a_{ij} = a_{ij}^{(k)} + \sum_{k=1}^{(k)} l_{i,k}\, a_{k,j}^{(k-1)} \; ; \quad \begin{array}{l} i = k+1, k+2, \dots, n \\ j = k, k+1, k+2, \dots, n \end{array}$$

This is nothing but $[A] = [L][U]$

or $$a_{ij}^{(k)} = a_{ij} - \sum_{m=1}^{k} l_{im}\, a_{mj}^{(m-1)}$$

where elements of $[L]$ are

$$l_{ij} = \begin{cases} l_{ik} & ; \text{ for } i > k \; ; \; k = 1, 2, 3, \ldots, n-1 \\ 1 & ; \text{ for } i = j \\ 0 & ; \text{ for } i < j \end{cases}$$

i.e.

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ l_{n1} & l_{n2} & l_{n3} & l_{n4} & \cdots & 1 \end{bmatrix}, \quad [U] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{bmatrix}$$

where $l_{21} = \dfrac{a_{21}^{(0)}}{a_{11}^{(0)}}$; $l_{32} = \dfrac{a_{32}^{(1)}}{a_{22}^{(1)}}$, etc.

So the steps are:

$$k = 1, 2, 3, \ldots, (n-1)$$

$$l_{kk} = 1 \; , \; l_{ik} = 0 \text{ for } i < k$$

$$u_{ij} = a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik}\, a_{ik}^{(k-1)}$$

$i = k+1, k+2, \ldots, n$

$j = k, k+1, k+2, \ldots, n$

$k = 1, 2, 3, 4, \ldots, n-1$

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} \quad ; \; i > k$$

So if you look into the example.

$$\begin{bmatrix} 80 & -20 & -20 \\ -20 & 40 & -20 \\ -20 & -20 & 130 \end{bmatrix} \quad R_2 = R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1 = R_2 - \left(\frac{-20}{80}\right) R_1 \qquad \text{Now } l_{21} = \frac{-1}{4}$$

$$R_3 = R_3 - \left(\frac{-20}{80}\right) R_1 \qquad l_{31} = \frac{-1}{4}$$

$$\Rightarrow \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & -25 & 125 \end{bmatrix} R_3 = R_3 - \left(\frac{-25}{35}\right) R_2 \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -1/4 & -\frac{5}{7} & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 80 & -20 & -20 \\ 0 & 35 & -25 \\ 0 & 0 & 750/7 \end{bmatrix}$$

Crout's Method

If in the decomposition $[A] = [L][U]$ the diagonal elements of $[U]$ are 1, then it is Crout's method.

$$[U] = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & & & & \\ 0 & 0 & \cdots & & 1 \end{bmatrix}$$

~~As an assignment~~ In Tutorial-2, I will be asking you to write the Crout's method described ~~for Doolittle's form~~ in the same form for Doolittle.

Cholesky Method

→ The LU decomposition for symmetric and positive definite matrix $[A]$ → efficiently computed if