Lecture 5: Gauss-Jordan Method

CE 601 NUMERICAL METHODS

LECTURE 5

01 - AUGUST - 2012

Yesterday we discussed on:

→ The Gauss elimination Method
- ↳ Forward Elimination
- ↳ Backward Substitution

→ The algorithm for elimination in any k^{th} step was:

$$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - l_{ik}\, a_{kj}^{(k-1)} \\ b_i^{(k)} &= b_i^{(k-1)} - l_{ik}\, b_k^{(k-1)} \end{aligned}\right\} \begin{aligned} i &= k+1, k+2, \dots, n \\ j &= k, k+1, k+2, \dots, n \\ k &= 1, 2, 3, \dots, n-1 \end{aligned}$$

where $l_{ik} = \dfrac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}$

Kindly check the website links for solved examples to see an example on Gauss elimination

Gauss - Jordan Method

→ The objective of Gauss - Jordan elimination is to reduce the matrix $[A]$ to an identity matrix.

→ In the process you will get a solution.

i.e.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Though a series of elimination the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & \cdots & & 0 & x_1 \\ 0 & 1 & 0 & \cdots & 0 & x_2 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & & 1 & x_n \end{array}\right]$$

i.e. $[A]\{x\} = \{b\}$ becomes

$[I]\{x\} = \{x\}$

→ The method is extension of Gauss-Elimination

* At each step,

(i) First, the pivot element is made unity

→ Dividing pivotal row with pivotal element.

i.e.
$$\left[\begin{array}{ccccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & & & & & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{kj} \cdots a_{kk} & \cdots & a_{kn} & b_k \\ \vdots & & & & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} \cdots a_{nk} & \cdots & a_{nn} & b_n \end{array}\right]$$

i.e. $$a_{kj}^{(k)} = \frac{a_{kj}^{(k-1)}}{a_{kk}^{(k-1)}} \quad ; \quad \begin{array}{l} j = k, k+1, k+2, \ldots, n \\ k = 1, 2, 3, \ldots, n \end{array}$$

$$\therefore a_{kk}^{(k)} = 1 \qquad \text{Also,} \quad b_k^{(k)} = \frac{b_k^{(k-1)}}{a_{kk}^{(k-1)}}$$

(ii) Second, row operations as described in Gauss elimination need to be done.

The multiplying factor now will be

$$l_{ik} = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k)}} = a_{ik}^{(k-1)} \quad ; \quad i = 1, 2, 3, \ldots k-1, k+1, k+2, \ldots, n$$

(iii) Above and below the major diagonal, the elements have to be zero. ∴ Row operations performed for all rows except the pivotal row.

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - a_{ik}^{(k-1)} a_{kj}^{(k-1)}$$

$i = 1, 2, 3, \ldots, k-1, k+1, k+2, \ldots, n$ (No k here)

$j = k, k+1, k+2, \ldots, n$

$$b_i^{(k)} = b_i^{(k-1)} - a_{ik}^{(k-1)} b_k \quad ; \quad i = 1, 2, 3, \ldots k-2, k-1, k+1, k+2, \ldots, n$$

(No 'k' here).

* In Gauss-Jordan elimination, there is no need of back-substitution as vector $\{b\}$ itself arrives to the solution.

We will quickly do a small example problem:

Problem

Solve the linear system

$$\begin{aligned} -5x_1 + 2x_2 + x_3 &= 2 \\ x_1 - 8x_2 + 3x_3 &= -6 \\ 3x_1 + x_2 - 7x_3 &= -16 \end{aligned}$$

using Gauss-Jordan elimination method.

Solution

The given linear system can be represented as $[A]\{x\} = b$

where $[A] = \begin{bmatrix} -5 & 2 & 1 \\ 1 & -8 & 3 \\ 3 & 1 & -7 \end{bmatrix}$; $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$; $\{b\} = \begin{Bmatrix} 2 \\ -6 \\ -16 \end{Bmatrix}$

The augmented matrix is $\left[\begin{array}{ccc|c} -5 & 2 & 1 & 2 \\ 1 & -8 & 3 & -6 \\ 3 & 1 & -7 & -16 \end{array}\right]$

First Step $(k=1)$

To make pivot element of $k=1$ row as 1.

$$\therefore a_{1j}^{(1)} = \frac{a_{1j}}{a_{11}} \quad ; \; j = 1, 2, 3$$

$$b_1^{(1)} = \frac{b_1}{a_{11}}$$

$\therefore$ The augmented matrix looks.

$$\left[\begin{array}{ccc|c} 1 & -0.400 & -0.200 & -0.400 \\ 1 & -8 & 3 & -6 \\ 3 & 1 & -7 & -16 \end{array}\right] \begin{array}{l} \\ R_2 = R_2 - a_{21} R_1 \\ R_3 = R_3 - a_{31} R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -0.400 & -0.200 & -0.400 \\ 0 & -7.600 & 3.200 & -5.600 \\ 0 & 2.200 & -6.400 & -14.800 \end{array}\right]$$

Second Step $(k=2)$

$$\left.\begin{aligned} a_{2j}^{(2)} &= \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \\ b_2^{(2)} &= \frac{b_2^{(1)}}{a_{22}^{(1)}} \end{aligned}\right\} \quad j = 1, 2, 3$$

$\therefore$ We get

$$\left[\begin{array}{ccc|c} 1 & -0.400 & -0.200 & -0.400 \\ 0 & 1 & -0.421 & 0.737 \\ 0 & 2.200 & -6.400 & -14.800 \end{array}\right] \begin{array}{l} R_1 = R_1 - a_{12} R_2 \\ \\ R_3 = R_3 - a_{32} R_2 \end{array}$$

$$a_{12} = -0.400$$
$$a_{32} = 2.200$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.368 & -0.105 \\ 0 & 1 & -0.421 & 0.737 \\ 0 & 0 & -5.474 & -16.421 \end{array}\right]$$

Third Step $(k=3)$

$$a_{3j}^{(3)} = \frac{a_{3j}^{(3)}}{a_{33}^{(2)}} \; ; \quad b_3^{(3)} = \frac{b_3^{(2)}}{a_{33}^{(2)}} \; ; \quad j = 1, 2, 3$$

$$\therefore \quad \left[\begin{array}{ccc|c} 1 & 0 & -0.368 & -0.105 \\ 0 & 1 & -0.421 & 0.737 \\ 0 & 0 & 1 & 3.000 \end{array}\right] \begin{array}{l} R_1 = R_1 - a_{13} R_3 \\ R_2 = R_2 - a_{23} R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore$ The solution vector is $\{x\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$