Lecture 44: Galerkin Weighted Residual method; The FEM; and The Course Conclusion

(14-Nov-2012)

LECTURE - 44
14-NOV-2012

FEM - Introduction

We have seen that to solve boundary-value ODE, you can utilise the following approach

$$\frac{d^2y}{dx^2} + Qy = F \quad ; \quad y(a) = y_a \;,\; y(b) = y_b$$

$$y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) + \cdots + C_I y_I(x)$$

where $y_i(x) \rightarrow$ Trial functions (usually polynomials)
$C_i \rightarrow$ Coefficients

These coefficients C_i can be evaluated using
→ Rayleigh - Ritz method (Method of calculus of variations)
→ Collocation Method (Residual formulations).

There is another method called to evaluate the coeffts C_i.

Galerkin Weighted Residual Method

⇒ This is also a residual method.

The steps are:

(i) Identify the concerned differential equation

$$\frac{d^2y}{dx^2} + Qy = F$$

(ii) Suggest approximate solution

$$y(x) \approx \tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_I y_I(x)$$

(iii) Define residual using approximate solution $\tilde{y}(x)$.

i.e. $R(x) = \frac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$

(iv) Choose the weighting functions $W_j(x)$; $j = 1, 2, 3, \ldots$

(v) Now set the integral of weighted residuals as zero

i.e. $\int_a^b W_j(x)\, R(x)\, dx = 0$

There will be j such integrals.

You have to define j according to number of unknown coefficient $C_1, C_2, C_3, \ldots, C_I$.

(vi) Solve these j-equation to get C_i's.

⇒ These three are methods that give approximate solutions to the entire domain of the BV-ODE.

→ You use polynomials as trial functions.

FEM for BV-ODE

If you recall in polynomial approximations, we have seen that rather than going for a higher degree polynomial for the entire domain it is better to use piece-wise polynomial approximation — that may yield better results.

The same philosophy is followed in the Approximate soln. method.

⇒ The approximate solution are obtained for small pieces of entire domain – called elements.

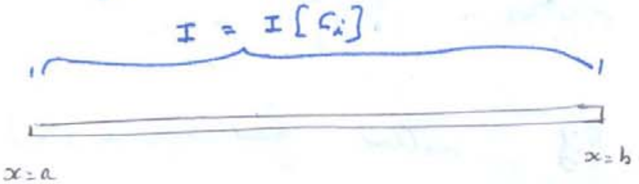
⇒ This becomes FEM.

You can use Rayleigh - Ritz
or
Galerkin Weighted Residual Method.

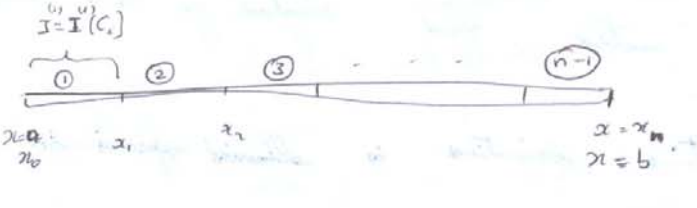
Please note:

→ If the variational functional of a problem is known in advance, you use Rayleigh-Ritz method to solve
(Usually in solid mechanics)

→ If the governing differential equation is known in advance, then it is better to use Galerkin weighted residual method. (Usually in fluid mechanics).



In the Rayleigh Ritz method



$$I = \sum_{j=1}^{n-1} I^{(j)}[c_i]$$

$i = 1, 2, 3, \ldots, I$

Steps:

(i) Formulate the problem.

eg: → If Rayleigh-Ritz method – find the functional to be extremised.

→ If Galerkin – Weighted Residual → differential equation and residual to be defined.

(ii) Discretise the global domain into sub-domains

(iii) The approximate solution for each element need to be defined.

$$\tilde{y}^{(1)}(x) = c_1 y_1^{(1)}(x) + c_2 y_2^{(1)}(x) + \cdots$$

$$\tilde{y}^{(2)}(x) =$$

$$\tilde{y}^{(3)}(x) = \cdots$$

$$\tilde{y}^{(n)}(x) = c_1 y_1^{(n)}(x) + c_2 y_2^{(n)}(x) + \cdots$$

(iv) In Rayleigh-Ritz method find $I[c_i]$ for each element.

In Galerkin method find residual for each element.

Please note that time derivative is ~~obtained~~ approximated using finite-difference.

OVERALL REVIEW OF THE COURSE CE 601

We have started this course with

→ Methods to Solve System of Linear Equations

- ★ Gauss Elimination
- ★ Gauss - Jordan
- ★ Doolittle Factorisation (LU)
- ★ Jacobi iteration
- ★ Gauss - Siedel
- ★ Successive Over Relaxation

→ Partially discussed methods to solve Eigen - value problems

→ To Solve Non-linear equations

- ★ Bisection method
- ★ Regula - Falsi method
- ★ ~~Fastest~~ Newton's method
- ★ Secant Method

→ Polynomial Approximations and Interpolations

- ★ Direct - Fit Polynomials
- ★ Lagrange Polynomials, etc.
- ★ Newton's polynomials
- ★ Splines
- ★ Method of least squares, etc.

→ Numerical Differentiation
- * Using polynomials
- * Taylor series
- * Difference formulas

→ Numerical Integration
- * Using Polynomials
- * Newton - Cotes Formulas
- * Gaussian Quadrature

→ Ordinary Differential Equations
- * Initial Value ODE
- * Boundary Value ODE
- * Apply FDM
  - ↳ Euler's method (Explicit & Implicit)
  - ↳ Modified " "
  - ↳ Runge - Kutta Method
  - ↳ Multi - point methods

→ Partial Differential Equations
- * FDM to solve. Elliptic PDE's
- * FDM to solve Parabolic PDE's
- * FDM to solve Hyperbolic PDE's
- * Criteria for stability, consistency, etc.

→ Brief Introduction to FEM
- * Rayleigh Ritz Method
- * Galerkin Weighted Residual Method.