Lecture 42: Rayleigh-Ritz Method; Collocation Method (12-Nov-2012)

LECTURE 43
12-NOV-2012

Introduction to FEM

In the last lecture we started the introduction to FEM

We were discussing on Rayleigh-Ritz method.

The functional

$$I[y] = \int_a^b G(x, y, y')\, dx$$

such that this functional should be extremum value.

Your ODE is $\frac{d^2y}{dx^2} + Py = F$ with fixed boundaries $x = a,\ y = y_a$; $x = b,\ y = y_b$

For the functional to be extremum

$$\delta I = 0$$

i.e. $$\int_a^b \left(\frac{\partial G}{\partial y} \delta y + \frac{\partial G}{\partial y'} \delta y' \right) dx = 0$$

i.e. $$\int_a^b \left(\frac{\partial G}{\partial y} \delta y + \frac{\partial G}{\partial y'} \delta\left(\frac{dy}{dx}\right) \right) dx = 0$$

or $$\int_a^b \left(\frac{\partial G}{\partial y} \delta y + \frac{\partial G}{\partial y'} \frac{d}{dx}(\delta y) \right) dx = 0$$

Integrating by parts:

$$\delta I = \int_a^b \left(\frac{\partial G}{\partial y} \delta y \right) dx + \frac{\partial G}{\partial y'} \delta y \Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) \delta y\, dx$$

You know $\delta y\,(x=a) = 0$
$\delta y\,(x=b) = 0$

$$\therefore \delta I = 0 = \int_a^b \left[\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) \right] \delta y\, dx$$

For any δy, $$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$$

This is Euler eqn. of calculus of variation.

$\therefore$ $G(x, y, y')$ has to be found in such a way that

$$\frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) \quad \text{gives the original ODE.}$$

Let us see the case below

$$G = (y')^2 - Qy^2 + 2Fy$$

$$I[y] = \int_a^b \left[(y')^2 - Qy^2 + 2Fy\right] dx$$

$$\frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)$$

i.e. $\frac{\partial}{\partial y}\left[(y')^2 - Qy^2 + 2Fy\right] = \frac{d}{dx}\left[\frac{\partial}{\partial y'}\left\{(y')^2 - Qy^2 + 2Fy\right\}\right]$

i.e. $-2Qy + 2F = \frac{d}{dx}(2y')$

i.e. $\frac{d^2y}{dx^2} + Qy = F \Rightarrow$ The reqd. ODE.

In approximate methods what we are doing is that the exact solution $y(x)$ is approximated by $\tilde{y}(x)$.

* $\tilde{y}(x) = \sum_{i=1}^{I} c_i y_i(x)$
* Determine $I[\tilde{y}(x)]$
* Determine $\frac{\partial I}{\partial c_i} = 0$
* Obtain c_i.

e.g. For $\frac{d^2y}{dx^2} + Qy = F$; $y(x_1) = y(0.0) = 0.0$

$y(x_2) = y(1.0) = Y$

Then $I[\tilde{y}] = \int_0^1 \left[(y')^2 - Q\tilde{y}^2 + 2F\tilde{y} \right] dx$

$$\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$
$$= C_1 x + C_2 x(x-1) + C_3 x^2(x-1)$$

So at $x = 0$, $y = 0.0$, we have.
and $x = 1.0$, $y = Y$

$C_1 = Y$.

$\therefore \tilde{y}(x) = Yx + C_2 x(x-1) + C_3 x^2(x-1)$

Obtain $I[\tilde{y}]$.

From that get $\frac{\partial I}{\partial C_2} = 0$

$\frac{\partial I}{\partial C_3} = 0$

Solve for C_2 & C_3.

Collocation Method

You need to evaluate residuals here.

The actual solution $y(x)$ of $\frac{d^2y}{dx^2} + Qy = F$

is approximated by $\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \dots C_I y_I(x)$

$\rightarrow$ Define $R(x) = \frac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$

This is actually $R(x, C_1, C_2, C_3, \dots, C_I)$

In I different values of 'x', we can set

$$R(x, C_1, C_2, \ldots C_I) = 0$$ to get I equations.

→ Solve the system of residual equations for getting coefficient $C_1, C_2, \ldots C_I$.

i.e. See the steps as follows:

In $\frac{d^2y}{dx^2} + Qy = F$; $y(0.0) = 0$, $y(1.0) = Y$

We have are taking

$$\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x)$$

Now $\tilde{y}(x) = Yx + C_2 x(x-1) + C_3 x^2(x-1)$

$$\Rightarrow R(x) = \frac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$$

$$= 2C_2 + C_3(6x-2) + QYx + QC_2 x(x-1) + QC_3 x^2(x-1) - F$$

We have C_2 and C_3 as unknowns.

Within the range from $x = 0$ to $x = 1$ substitute any x to obtain corresponding $R(x) = 0.0$

At $(x = 1/3) \Rightarrow R(\frac{1}{3}) = 0$

$(x = 2/3) \Rightarrow R(2/3) = 0$

From these two equations get C_2 & C_3.

Galerkin Weighted Residual Method

* This is also a residual method.
* Galerkin weighting residual method — obtained by integrating the residual over the domain of interest.

$\int W_j(x) \, R(x) \, dx = 0$

The steps are:

(i) The differential equation say $\frac{d^2y}{dx^2} + Qy = F$

(ii) $\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_I y_I(x)$

(iii) $R(x) = \frac{d^2\tilde{y}}{dx^2} + Q\tilde{y} - F$

(iv) Choose weighting functions $W_j(x)$, $j = 1, 2, 3, \ldots$

(v) Set the integral of weighted residuals as zero.

$$\int_{x_1}^{x_2} W_j(x) \, R(x) \, dx = 0 \quad ;$$

Depending on the number of coeffts $C_1, C_2, C_3 \ldots$, we require that many number of weighing factors $W_j(x)$.

Quiz

Find the finite-difference equation for the PDE

$$\frac{\partial^2 T}{\partial t^2} = \beta \frac{\partial^2 T}{\partial x^2}.$$

Find the criteria parameter in the FDE.