Lecture 42: Introduction to FEM (07-Nov-2012)

LECTURE 42
07-NOV-2012

SOLVING DIFFERENTIAL EQUATIONS

We have seen two different categories of differential equations that are used to mathematically describe the various features in engineering and sciences

→ Ordinary Differential Equations
* IV - ODE
* BV - ODE

→ Partial Differential Equations
* Elliptic PDE
* Parabolic PDE
* Hyperbolic PDE

These differential equations were solved using
* Finite- Difference Method

till now.

Today, we will briefly go through Finite - Element Methods to solve such differential equations

* FEM is a vast topic and itself form a separate course.
* With meagre two hours of lecture on FEM, the various features in it may not be possible to explain or solve.
* However, our objective here is to briefly cover some introductory aspects on FEM ~~for~~
* You may be just provided ~~b~~ a hint that like FDM, one may also use FEM for solving

differential equations.

* For more knowledge or information on FEM you may pursue higher level courses or books.

Q: What was FDM doing in solving differential equations?

S:→ FDM works on a methodology in which

→ derivatives approximated by finite-difference formulas
→ substitute these formulas in original differential eqn.
→ obtain algebraic ~~a~~ equations that need to be solved.

Now, the exact solution of the differential equation can be given ~~approximated~~ by approximate solutions that are linear combination of specific trial function

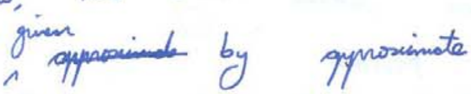
* The trial functions – independent functions
– satisfy boundary conditions.

* There will be unknown coefficient or weightage that have to be assigned for each trial function

Consider a 1D BV-ODE

$$\frac{d^2y}{dx^2} + \Phi y = F$$

with say
$y(x_1) = y_1$
$y(x_2) = y_2$



Let $\Phi = \Phi(x)$
$F = F(x)$

The exact solution of the above BV-ODE is $y(x)$.

This exact solution is approximated by say $\tilde{y}(x)$

$$\tilde{y}(x) = \sum_{i=1}^{I} C_i\, y_i(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots C_I y_I(x)$$

where $y_i \rightarrow$ trial functions
$C_i \rightarrow$ weightages

$$y(x) \approx \tilde{y}(x) = \sum_{i=1}^{I} C_i\, y_i(x)$$

$\Rightarrow$ Now if this entire domain is divided into small elements
$\rightarrow$ We can obtain the solution of such each individual elements and then clubbed to form the global solution.
This is FEM.

<u>Rayleigh - Ritz Method</u>

$\rightarrow$ Calculus of Variations is used in this approach.
$\rightarrow$ Objective of calculus of variations – to maximise or minimise functionals (function of functions).

$\rightarrow$ In the BV-ODE $\quad \dfrac{d^2y}{dx^2} + Py = F$

$y(x_1) = y_1$
$y(x_2) = y_2$

You know the exact solution is $y(x)$..

→ A simple problem in calculus of variations is about a function in one independent variable $y(x)$ is a function in x.

→ We need to extremise a functional

Consider the functional

$$I[y] = \int_a^b G(x, y, y')\, dx$$

where $I[y]$ is the functional that has to be extremised in calculus of variations.

$G(x, y, y')$ → fundamental function

'a' and 'b' are the limits in x-direction.

→ The objective here is to determine that particular $y(x)$ which extremises $I[y]$.

→ Derivative of a functional → called <u>Variation</u>.

Use the symbol δ

* Derivative of ordinary function given by symbol 'd'.

As we are having fixed end points 'a' and 'b'.

$$\delta I = \int_a^b \left(\frac{\partial G}{\partial y}\, \delta y + \frac{\partial G}{\partial y'}\, \delta y' \right) dx$$

$$= \int_a^b \left(\frac{\delta G}{\delta y}\, \delta y + \frac{\partial G}{\partial y'}\, \delta\left(\frac{dy}{dx}\right) \right) dx$$

$$= \int_a^b \left(\frac{\delta G}{\delta y}\, \delta y + \frac{\partial G}{\partial y'} \frac{d}{dx}(\delta y) \right) dx$$

Integrating by parts

$$\delta I = \int_a^b \left(\frac{\partial G}{\partial y}\, \delta y \right) dx + \frac{\partial G}{\partial y'}\, \delta y \Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) \delta y \, dx$$

Now for fixed points $\delta y\,(x=a) = 0$
$\delta y\,(x=b) = 0$

$$\therefore \delta I = \int_a^b \left[\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) \right] \delta y \, dx = 0$$

(Extremisation).

For any δy, then

$$\boxed{\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right) = 0} \quad \text{or} \quad \frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial G}{\partial y'} \right)$$

This is Euler's equation of calculus of variation

So in the given problem the fundamental function G $G(x, y, y')$ has to be formulated in such a way that $I[y] = \int_a^b G(x, y, y')\, dx$

As $I[y]$ has to be extremum

Let us correlate our differential equation $\frac{d^2y}{dx^2} + Qy = F$ to this functional $I[y]$ by suggesting that the extremisation

~~$\therefore I[y] = y'' + Qy = F$~~

of $I[y]$ gives $\frac{d^2y}{dx^2} + Qy - F = 0$

i.e. $\delta I = 0 = y'' + \Phi y - F$

→ The fundamental function G for this case will be

$$G(x, y, y') = (y')^2 - \Phi y^2 + 2Fy$$

$$\therefore I[y] = \int_a^b \left[(y')^2 - \Phi y^2 + 2Fy\right] dx$$

We know

$$\frac{\partial G}{\partial y} = \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)$$

i.e. $\frac{\partial}{\partial y}\left[(y')^2 - \Phi y^2 + 2Fy\right] = \frac{d}{dx}\left\{\frac{\partial}{\partial y'}\left((y')^2 - \Phi y^2 + 2Fy\right)\right\}$

i.e. $-2\Phi y + 2F = \frac{d}{dx}(2y') = 2\frac{d^2y}{dx^2}$

i.e. $\underline{\frac{d^2y}{dx^2} + \Phi y = F}$ The required ODE

Now this exact solution is approximated by $\tilde{y}(x)$

$$\tilde{y}(x) = \tilde{y}(x, c, d, e, f, \ldots)$$

$$I[y(x)] \approx I[\tilde{y}] = I[y(x, c, d, e, f, \ldots)]$$

$$\delta I = 0 = \frac{\partial I}{\partial c}\delta c + \frac{\partial I}{\partial d}\delta d + \cdots$$

This is satisfied only if $\frac{\partial I}{\partial c} = 0$, $\frac{\partial I}{\partial d} = 0, \ldots$

∴ To determine the function $\tilde{y}(x)$ you require parameter values $c, d, e, f, \ldots$

Once these parameters are identified you get $\tilde{y}(x)$, which is approximate soln. of BV-ODE.

So the Rayleigh - Ritz method Steps:

(i) Determine $I[y(x)]$ that gives the BV-ODE from the Euler equation

(ii) $y(x) \approx \tilde{y}(x) = \sum_{i=1}^{I} C_i \, y_i(x)$

(iii) Substitute $\tilde{y}(x) = \sum_{i=1}^{I} C_i \, y_i$ in $I[y(x)]$

Get $I[C_i]$

(IV) $\dfrac{\partial I}{\partial C_i} = 0$

(V) Solve to get C_i

e.g. If $y(x_1) = y(0.0) = 0.0$

$y(x_2) = y(1.0) = Y$

Then $$I[\tilde{y}] = \int_0^1 \left[(y')^2 - Q y^2 + 2Fy \right] dx$$

$$y(x) = C_1 \, y_1(x) + C_2 \, y_2(x) + C_3 \, y_3(x)$$
$$= C_1 x + C_2 \, x(x-1) + C_3 \, x^2 (x-1)$$

The three trial functions $y_1(x)$, $y_2(x)$ and $y_3(x)$ are linearly independent.

Applying boundary conditions $y_1(x = 0.0) =$

$y(1.0) = Y = C_1$

$\therefore\ y(x) = Yx + C_2\, x(x-1) + C_3\, x^2(x-1) = y(x, C_2, C_3)$

Now integrate

$$I[y] = \int_0^1 \left((y')^2 - Qy^2 + 2Fy \right) dx$$

Obtain $\dfrac{\partial I}{\partial C_2} = 0$, $\dfrac{\partial I}{\partial C_3} = 0$ to obtain C_2 and C_3.