Lecture 41: Hyperbolic Partial Differential Equations (06-Nov-2012)

LECTURE 41
06-NOV-2012

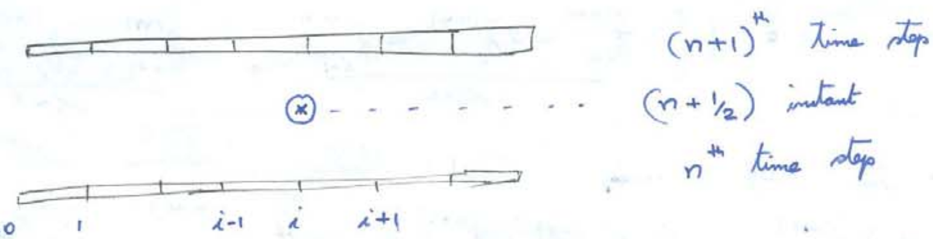
Parabolic ~~Hyperbolic~~ Partial Differential Equations

* Yesterday, we have seen the FTCS method and BTCS method for solving the parabolic PDE (Diffusion equation)

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

* FTCS method gave explicit expressions of FDE
* BTCS gave implicit expressions of FDE
* In both FTCS and BTCS the time derivative $\frac{\partial f}{\partial t}$ was approximated by a first-order finite-difference formula.
  → That means that the order of approximation for time was less.
* We can have a second-order finite-difference expression for time.
* Then the corresponding FDE for diffusion equation will be as follows:

Crank-Nicolson Method



As suggested yesterday, the time domain and space domain are continuous for the given problem:

That is the given diffusion equation should be satisfied at any given instant of time, as well as it should be satisfied at any point in the bounded domain.

Therefore one can easily visualize that

$$\frac{\partial f}{\partial t}\bigg|_i^{(n+1/2)} = \alpha \frac{\partial^2 f}{\partial x^2}\bigg|_i^{(n+1/2)}$$

Let us keep the information $f_i^{n+1/2}$ as the base point

Using Taylor's series:

$$f_i^{(n+1)} = f_i^{(n+1/2)} + \frac{\partial f}{\partial t}\bigg|_i^{(n+1/2)}\left(\frac{\Delta t}{2}\right) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\bigg|_i^{n+1/2}\frac{\Delta t^2}{4} + \cdots\cdots$$

$$f_i^{(n)} = f_i^{(n+1/2)} + \frac{\partial f}{\partial t}\bigg|_i^{(n+1/2)}\left(\frac{-\Delta t}{2}\right) + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\bigg|_i^{n+1/2}\frac{\Delta t^2}{4} + \cdots\cdots$$

$$\therefore f_i^{(n+1)} - f_i^{(n)} = \frac{\partial f}{\partial t}\bigg|_i^{(n+1/2)}\left[\frac{\Delta t}{2} + \frac{\Delta t}{2}\right] + 0 + O(\Delta t^3)$$

$$\therefore \frac{\partial f}{\partial t}\bigg|_i^{(n+1/2)} = \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} + O(\Delta t^2)$$

$$\frac{\partial^2 f}{\partial x^2}\bigg|_i^{(n+1/2)} = \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}\bigg|_i^{(n+1)} + \frac{\partial^2 f}{\partial x^2}\bigg|_i^{(n)}\right] \cancel{+ O\,\Delta x}$$

$$= \frac{1}{2}\left[\frac{f_{i+1}^{(n+1)} - 2f_i^{(n+1)} - f_{i-1}^{(n+1)}}{\Delta x^2} + \frac{f_{i+1}^{(n)} - 2f_i^{(n)} + f_{i-1}^{(n)}}{\Delta x^2}\right] + O(\Delta x^2)$$

$\therefore$ The FDE using Crank - Nicolson Scheme will be:

$$\left(\frac{\alpha\Delta t}{2\Delta x^2}\right)f_{i-1}^{(n+1)} - \left(1 + \frac{\alpha\Delta t}{\Delta x^2}\right)f_i^{(n+1)} + \frac{\alpha\Delta t}{2\Delta x^2}f_{i+1}^{(n+1)} = -\left(\frac{\alpha\Delta t}{2\Delta x^2}\right)f_{i-1}^{(n)} + \left(-1 + \frac{\alpha\Delta t}{\Delta x^2}\right)f_i^{(n)} - \frac{\alpha\Delta t}{2\Delta x^2}f_{i+1}^{(n)}$$

Like diffusion equation $\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$
we have another classical equation called Convective-Diffusion equation (or Advective-Dispersive Equation).

$$\boxed{\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = D \frac{\partial^2 f}{\partial x^2}}$$

where $u \rightarrow$ convection velocity
$\alpha \rightarrow$ Diffusion coefficient.

$\Rightarrow$ You can use the FTCS, or BTCS, or Crank-Nicolson scheme to solve this equation as well.

Hyperbolic Partial Differential Equations

Common hyperbolic PDE's seen in engineering and science problem are:

Convection equation $\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0$

Wave equation $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$

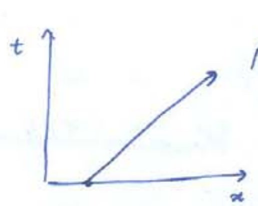
Let us consider the convection equation for fluid flow

$\frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} = 0$, $u \rightarrow$ convection velocity

The location of a fluid particle that is moving in such problems can be given as $x(t)$

and $\frac{dx}{dt} = u$

The pathline of the fluid particle
$$x = x_0 + \int_{t_0}^{t} u(t)\, dt$$



$$x = x_0 + \int_{t_0}^{t} u(t)\,dt$$

Along the pathline, the convection equation is:

$$\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} = 0$$

i.e. $$\frac{\partial \phi}{\partial t} + \frac{dx}{dt}\frac{\partial \phi}{\partial x} = 0$$

i.e. $$\frac{d\phi}{dt} = 0$$

or $$\boxed{\phi = \text{constant}}$$

$\Rightarrow$ The fluid particle $\phi \to$ convected along the pathline. (Characteristic path).

$\Rightarrow$ One may attempt to solve by identifying characteristic paths. We are not going to discuss them here.

$\Rightarrow$ Consider a general quasi-linear first-order PDE

$$a\frac{\partial \phi}{\partial t} + b\frac{\partial \phi}{\partial x} = c$$

As $\phi \to \phi(t, x)$

we can have $$d\phi = \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x}dx$$

As $$c = \frac{\partial \phi}{\partial t}a + \frac{\partial \phi}{\partial x}b$$

i.e. $$\begin{bmatrix} a & b \\ dt & dx \end{bmatrix}\begin{Bmatrix} \partial\phi/\partial t \\ \partial\phi/\partial x \end{Bmatrix} = \begin{Bmatrix} c \\ d\phi \end{Bmatrix}$$

$\Rightarrow$ There can be unique values for $\partial\phi/\partial t$ and $\partial\phi/\partial x$ for non-zero determinant of $\begin{bmatrix} a & b \\ dt & dx \end{bmatrix}$

$\Rightarrow$ If det is zero, then $a\,dx - b\,dt = 0$

$$a\,dx - b\,dt = 0$$

Or. $\boxed{\dfrac{dx}{dt} = \dfrac{b}{a}}$

→ This is differential equation for a family of paths.
'a' and 'b' are real functions.
→ ∴ Single quasi-linear first order PDE is considered hyperbolic. (eg. Convection equation).

Features of Convection

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \quad ; \quad u \to \text{convection velocity}$$

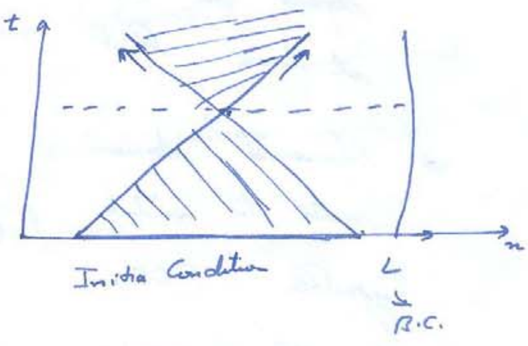
Solution $f(x,t) = F(x-ut)$

$f(x,0) = \phi(x)$

i.e. $F(x) = \phi(x)$

∴ $F(x-ut) = \phi(x-ut)$

$$\frac{dx}{dt} = u$$

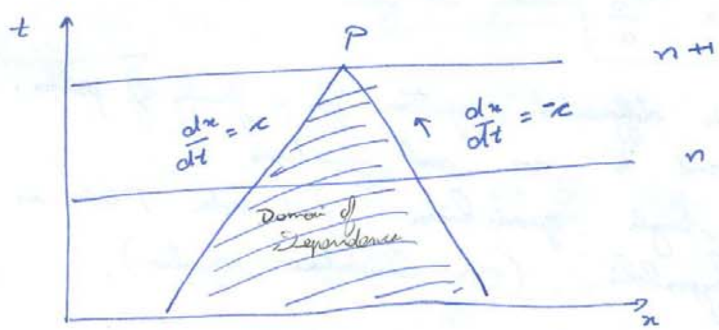


The Finite-Difference Method

→ Discretising
→ Approximating
→ Substituting
→ Solving

Objective of FDE method to march the solution of a hyperbolic PDE from level n to level n+1.

There is a physical domain of dependence of a hyperbolic PDE



$c = \frac{dx}{dt}$ → Physical Information Propagation speed.

∴ Solution at point P at time level (n+1) should not depend on the solution at any of the other points at time level (n+1). $c_n = \frac{\Delta x}{\Delta t}$.

⇒ Numerical domain of dependence of explicit FDM suited to match physical domain of dependence of hyperbolic PDE's.

⇒ Hyperbolic PDE's - Therefore solved by explicit methods.

i.e. $\cancel{\left.\frac{\partial f}{\partial x}\right|_i^{(n)}} = \cancel{f_i}$

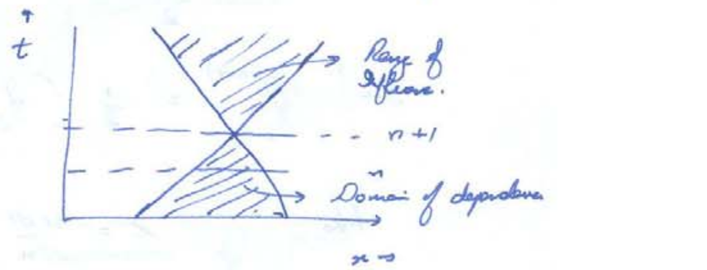
$\frac{\partial f}{\partial x}$ → Shows the physical convection

In the convection equation, the characteristic paths are the pathlines given by

$$\frac{dx}{dt} = u$$

Physical information propagation speed = u

* Solution at a point depends only on the information in the domain of dependence.
* Also solution at a point influences the solution in the range of influence



∴ We may have to use
→ First-order spatial derivative (Backward or Forward).
→ These are Upwind approximation.

$$\left.\frac{\partial f}{\partial x}\right|_i^{(n)} = \frac{f_i^{(n)} - f_{i-1}^{(n)}}{\Delta x} \quad \text{(Backward)}$$

$$\left.\frac{\partial f}{\partial x}\right|_i^{(n)} = \frac{f_{i+1}^{(n)} - f_i^{(n)}}{\Delta x} \quad \text{(Forward)}$$

However you can also go for centered-difference method

$$\left.\frac{\partial f}{\partial x}\right|_i^{(n)} = \frac{f_{i+1}^{(n)} - f_{i-1}^{(n)}}{2\Delta x}$$

FTCS Method

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$$

$$\frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} + u\,\frac{f_{i+1}^{(n)} - f_{i-1}^{(n)}}{2\Delta x} = 0$$

i.e. $$f_i^{(n+1)} = f_i^{(n)} - \frac{1}{2}\left(u\frac{\Delta t}{\Delta x}\right)\left(f_{i+1}^{(n)} - f_{i-1}^{(n)}\right)$$

$$c = \frac{u\Delta t}{\Delta x} \rightarrow \text{Convection number}$$

Lax Method

$$\phi_i^{(n)} \approx \frac{\phi_{i+1}^{(n)} + \phi_{i-1}^{(n)}}{2}$$

Then $$\phi_i^{(n+1)} = \frac{1}{2}\left(\phi_{i+1}^{(n)} + \phi_{i-1}^{(n)}\right) - \frac{c}{2}\left(\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}\right)$$

Where $$c = \frac{u\Delta t}{\Delta x} \leq 1$$

Lax method is explicit

Lax - Wendroff One step Method

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = 0$$

$$\phi_i^{(n+1)} = \phi_i^{(n)} + \frac{\partial \phi}{\partial t}\Big|_i^{(n)} \Delta t + \frac{1}{2}\frac{\partial^2 \phi}{\partial t^2}\Big|_i^{(n)} \Delta t^2 + \cdots$$

$\frac{\partial \phi}{\partial t} = -u\frac{\partial \phi}{\partial x}$

Two spatial derivative $$\frac{\partial \phi}{\partial x}\Big|_i^{(n)} = \frac{\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}}{2\Delta x}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial}{\partial t}\left(-u\frac{\partial \phi}{\partial x}\right) = u^2\frac{\partial^2 \phi}{\partial x^2} \qquad \frac{\partial^2 \phi}{\partial x^2}\Big|_i^{(n)} = \frac{\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}}{\Delta x^2}$$

$$\therefore \phi_i^{(n+1)} = \phi_i^{(n)} - u\left(\frac{\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}}{2\Delta x}\right)\Delta t + \frac{1}{2}u^2\left(\frac{\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}}{\Delta x^2}\right)\Delta t^2$$

$$\phi_i^{(n+1)} = \phi_i^{(n)} - u\left(\frac{\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}}{2\Delta x}\right)\Delta t + \frac{1}{2}u^2\left(\frac{\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}}{\Delta x^2}\right)\Delta t^2$$

Using convection number $c = \frac{u\Delta t}{\Delta x}$

$$\phi_i^{(n+1)} = \phi_i^{(n)} - \frac{c}{2}\left(\phi_{i+1}^{(n)} - \phi_{i-1}^{(n)}\right) + \frac{c^2}{2}\left(\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}\right)$$

$c = \frac{u\Delta t}{\Delta x} \leq 1.0$ is the condition required.