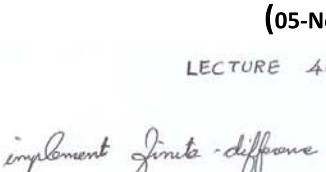
# Lecture 40: Parabolic Partial Differential Equations (05-Nov-2012)

PARTIAL DIFFERENTIAL EQUATIONS LECTURE 40

Yesterday we were discussing on how to implement finite-difference methods for solving PDEs in non-rectangular domains

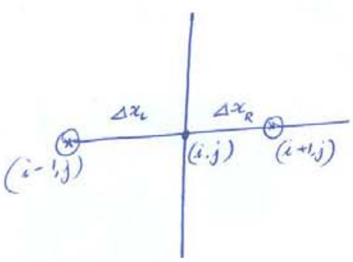
Some approaches for modelling non-rectangular domains



* Approximating physical domain
* Use other co-ordinate system
* Non-uniform finite-difference approximations
* Transform the non-rectangular space to rectangular space.

→ You can use cylindrical coordinates here.

## The Non-uniform FD Approximations




If the grid space between grid points are non-uniform, then the FD method should be given as below.

$$\phi_{i+1,j} = \phi_{i,j} + \frac{\partial \phi}{\partial x}\Big|_{i,j} \Delta x_R + \frac{\partial^2 \phi}{\partial x^2}\Big|_{i,j} \frac{(\Delta x_R)^2}{2} + \frac{1}{6}(\Delta x_R)^3 \frac{\partial^3 \phi}{\partial x^3}\Big|_{(i,j)} + \cdots \quad (1)$$

$$\phi_{i-1,j} = \phi_{i,j} + \frac{\partial \phi}{\partial x}\Big|_{(i,j)} (-\Delta x_L) + \frac{\partial^2 \phi}{\partial x^2}\Big|_{(i,j)} (\Delta x_L)^2 + \frac{1}{6}(-\Delta x_L)^3 \frac{\partial^3 \phi}{\partial x^3}\Big|_{(i,j)} + \cdots \quad (2)$$

$\therefore$ Multiplying by $\Delta x_L$ on eqn (1) and $\Delta x_R$ on eq (2) and adding:

$$\Delta x_L \phi_{i+1,j} + \Delta x_R \phi_{i-1,j} = (\Delta x_L + \Delta x_R)\phi_{i,j} + \frac{1}{2}\left(\Delta x_L \Delta x_R^2 + \Delta x_L^2 \Delta x_R\right)\frac{\partial^2 \phi}{\partial x^2}\Big|_{(i,j)} + \cdots$$

or

$$\frac{\partial^2 \phi}{\partial x^2}\Big|_{(i,j)} = \frac{2\Delta x_L}{(\Delta x_L \Delta x_R^2 + \Delta x_L^2 \Delta x_R)}\phi_{i+1,j} - \frac{2(\Delta x_L + \Delta x_R)}{(\Delta x_L \Delta x_R^2 + \Delta x_L^2 \Delta x_R)}\phi_{i,j} + \frac{2\Delta x_R}{(\Delta x_L \Delta x_R^2 + \Delta x_L^2 \Delta x_R)}$$

Similarly you can get $\frac{\partial^2 \phi}{\partial y^2}\Big|_{(i,j)}$

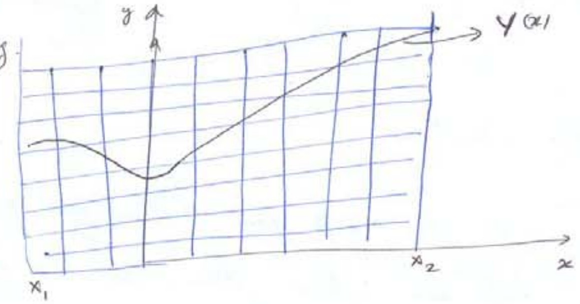
## Grid Transformation

* Most of the engineering and science problems, described in PDE's are usually in orthogonal Cartesian coordinate system.

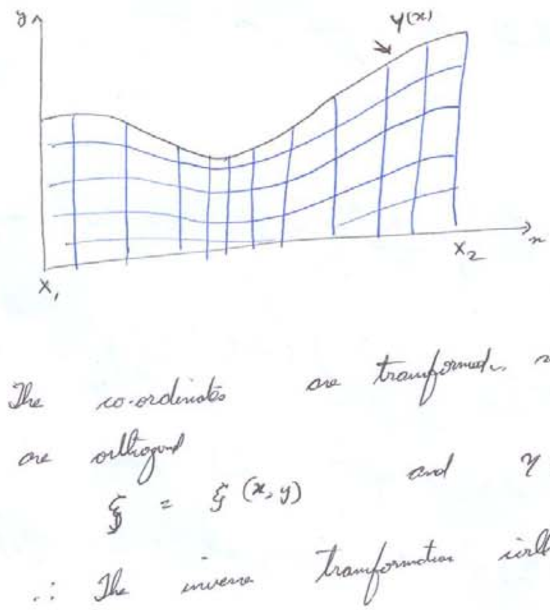
* When you used the finite-difference method, the discretised computational domain are also orthogonal in nature.

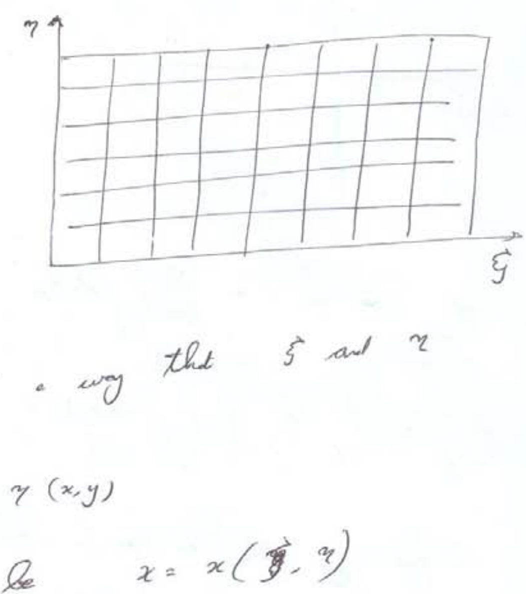
→ The non-rectangular physical domain can be converted to rectangular computational domain by grid transformation.

eg.




→ In the figure shown the upper boundary of the physical space does not fall on grid cell boundaries.




The co-ordinates are transformed such a way that $\xi$ and $\eta$ are orthogonal

$$\xi = \xi(x, y) \quad \text{and} \quad \eta = \eta(x, y)$$

∴ The inverse transformation will be $x = x(\xi, \eta)$

eg. In $a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = c$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x}$$

Similarly for $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$

At $x = X_1$, $\xi = \xi_{min}$
$x = X_2$, $\xi = \xi_{max}$

$\frac{\partial \xi}{\partial x}$ and $\frac{\partial \xi}{\partial y}$ are properties, $\therefore$ $\xi = \xi(x, y)$

## PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Consider the diffusion equation

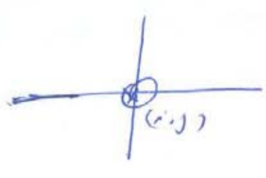
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

or $\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$ , etc.

⇒ These are propagation problems.
⇒ Propagation problems require initial values.

At any grid-point $(i, j)$ the partial derivatives will be

$$\frac{\partial f}{\partial t} = \alpha\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$




$\left.\frac{\partial f}{\partial t}\right|_{(i,j)}^{(n)}$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)}^{(n)}$ , $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(i,j)}^{(n)}$

i.e. $\left.\frac{\partial f}{\partial t}\right|_{(i,j)}^{(n)} = \frac{f_{i,j}^{(n+1)} - f_{i,j}^{(n)}}{\Delta t}$ → Forward Time.

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j}^{(n)} = \frac{f_{i+1,j}^{(n)} - 2f_{i,j}^{(n)} + f_{i-1,j}^{(n)}}{\Delta x^2}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(i,j)}^{(n)} = \frac{f_{i,j+1}^{(n)} - 2f_{i,j}^{(n)} + f_{i,j-1}^{(n)}}{\Delta y^2}$$

One can develop Forward Time Centered Space (FTCS) finite difference equation for parabolic PDE.

$$\frac{T_{i,j}^{(n+1)} - T_{i,j}^{(n)}}{\Delta t} = \alpha \left[ \frac{T_{i+1,j}^{(n)} - 2T_{i,j}^{(n)} + T_{i-1,j}^{(n)}}{\Delta x^2} + \frac{T_{i,j+1}^{(n)} - 2T_{i,j}^{(n)} + T_{i,j-1}^{(n)}}{\Delta y^2} \right]$$

i.e.
$$T_{i,j}^{(n+1)} = T_{i,j}^{(n)} + \left(\alpha \frac{\Delta t}{\Delta x^2}\right) T_{i+1,j}^{(n)} - 2\alpha\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) T_{i,j}^{(n)} + \left(\frac{\alpha\Delta t}{\Delta x^2}\right) T_{i-1,j}^{(n)} + \left(\frac{\alpha\Delta t}{\Delta y^2}\right) T_{i,j+1}^{(n)} + \frac{\alpha\Delta t}{\Delta y^2} T_{i,j-1}^{(n)}$$

$\Rightarrow$ Explicit Method.

$\rightarrow$ You can check the consistency and stability of the method.

Do the stability analysis of one-dimensional FTCS finite-difference equation of $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

$$T_i^{(n+1)} = T_i^{(n)} + \left(\alpha \frac{\Delta t}{\Delta x^2}\right) T_{i+1}^{(n)} - 2\frac{\alpha \Delta t}{\Delta x^2} T_i^{(n)} + \left(\frac{\alpha\Delta t}{\Delta x^2}\right) T_{i-1}^{(n)}$$

Let us define $d = \frac{\alpha \Delta t}{\Delta x^2}$

$$\therefore T_i^{(n+1)} = T_i^{(n)} + d\left(T_{i+1}^{(n)} - 2T_i^{(n)} + T_{i-1}^{(n)}\right)$$

$$T_i^{(n+1)} = G\, T_i^{(n)}$$

where $G \rightarrow$ Amplification factor

$|G| \leq 1.0$ for stability

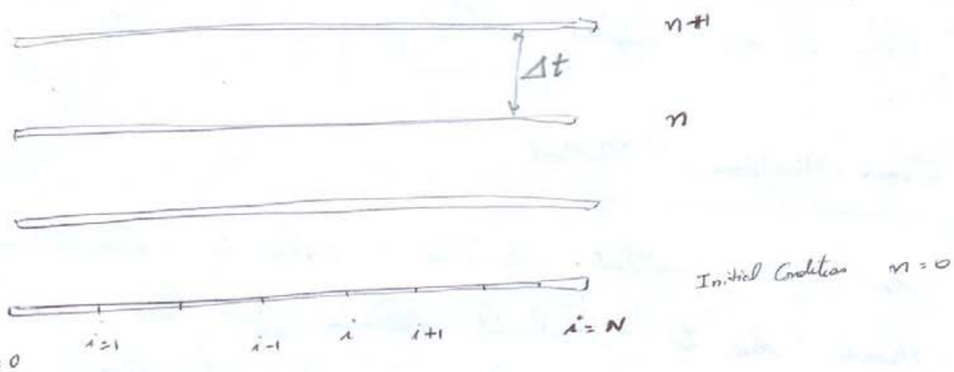
## Backward Time Centered Space Method (BTCS)

Consider the one-dimensional diffusion equation

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \rightarrow (1)$$

If we approximate the derivative at $(n+1)$ time step

Please note your problem domain (temporal and spatial)




The one-dimensional spatial domain has to be evaluated at each time step

Let $\Delta t$ → Time step size

$n$ → The time step at present

$n = 0$ → Initial condition

$n + 1$ → Future time step.

$$\therefore \left.\frac{\partial f}{\partial t}\right|_i^{(n+1)} = \frac{f_i^{(n+1)} - f_i^{(n)}}{\Delta t} \qquad O(\Delta t) \qquad \text{Backward-Difference Approx.}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i^{(n+1)} = \frac{f_{i+1}^{(n+1)} - 2 f_i^{(n+1)} + f_{i-1}^{(n+1)}}{\Delta x^2} \qquad O(\Delta x^2)$$

Substituting in eqn. (1)

$$\frac{\zeta_i^{(n+1)} - \zeta_i^{(n)}}{\Delta t} = \alpha \, \frac{\zeta_{i+1}^{(n+1)} - 2\zeta_i^{(n+1)} + \zeta_{i-1}^{(n+1)}}{\Delta x^2}$$

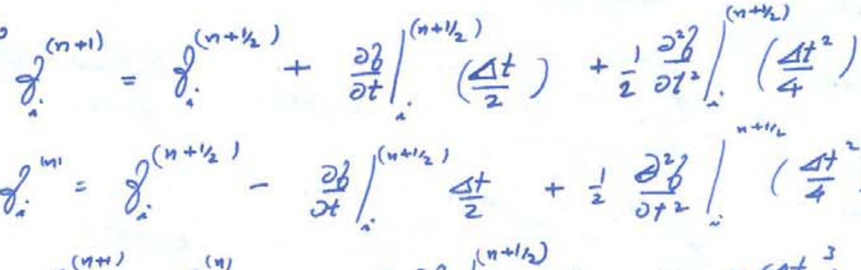
or $$\left(\frac{\alpha \Delta t}{\Delta x^2}\right) \zeta_{i-1}^{(n+1)} - \left(1 + 2\frac{\alpha \Delta t}{\Delta x^2}\right)\zeta_i^{(n+1)} + \left(\frac{\alpha \Delta t}{\Delta x^2}\right)\zeta_{i+1}^{(n+1)} = -\zeta_i^{(n)}$$

This will be the FDE for diffusion pde when we use BTCS method.

This is an implicit method.

## Crank - Nicolson Method

The BTCS method described earlier is unconditionally stable. However due to backward difference for time, we have first-order approximation for temporal derivative.




Now
$$\zeta_i^{(n+1)} = \zeta_i^{(n+1/2)} + \frac{\partial \zeta}{\partial t}\Big|_i^{(n+1/2)} \left(\frac{\Delta t}{2}\right) + \frac{1}{2}\frac{\partial^2 \zeta}{\partial t^2}\Big|_i^{(n+1/2)} \left(\frac{\Delta t^2}{4}\right) + \cdots$$

$$\zeta_i^{(n)} = \zeta_i^{(n+1/2)} - \frac{\partial \zeta}{\partial t}\Big|_i^{(n+1/2)} \frac{\Delta t}{2} + \frac{1}{2}\frac{\partial^2 \zeta}{\partial t^2}\Big|_i^{(n+1/2)} \left(\frac{\Delta t^2}{4}\right) + \cdots$$

$$\therefore \zeta_i^{(n+1)} - \zeta_i^{(n)} = 2\frac{\partial \zeta}{\partial t}\Big|_i^{(n+1/2)} \frac{\Delta t}{2} + O(\Delta t^3)$$

$$\therefore \left.\frac{\partial \phi}{\partial t}\right|_i^{n+1/2} = \frac{\phi_i^{(n+1)} - \phi_i^{(n)}}{\Delta t} + O(\Delta t^2)$$

This is second-order approximation.

$$\therefore \left.\frac{\partial \phi}{\partial t}\right|_i^{n+1/2} = \alpha \left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{n+1/2}$$

Now $\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{n+1/2} = \frac{1}{2}\left[\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{(n+1)} + \left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{(n)}\right]$

$$\text{i.e.}\quad \frac{\phi_i^{(n+1)} - \phi_i^{(n)}}{\Delta t} = \frac{\alpha}{2}\left[\left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{(n+1)} + \left.\frac{\partial^2 \phi}{\partial x^2}\right|_i^{(n)}\right]$$

$$\text{i.e.}\quad \frac{\phi_i^{(n+1)} - \phi_i^{(n)}}{\Delta t} = \frac{\alpha}{2}\left[\frac{\phi_{i+1}^{(n+1)} - 2\phi_i^{(n+1)} + \phi_{i-1}^{(n+1)}}{\Delta x^2} + \frac{\phi_{i+1}^{(n)} - 2\phi_i^{(n)} + \phi_{i-1}^{(n)}}{\Delta x^2}\right]$$

or

$$\boxed{\begin{aligned}\left(\frac{\alpha\Delta t}{2\Delta x^2}\right)\phi_{i-1}^{(n+1)} - \left(1 + \frac{\alpha\Delta t}{\Delta x^2}\right)\phi_i^{(n+1)} + \frac{\alpha\Delta t}{2\Delta x^2}\phi_{i+1}^{(n+1)} \\ = -\left(\frac{\alpha\Delta t}{2\Delta x^2}\right)\phi_{i-1}^{(n)} + \left(-1 + \frac{\alpha\Delta t}{\Delta x^2}\right)\phi_i^{(n)} - \frac{\alpha\Delta t}{2\Delta x^2}\phi_{i+1}^{(n)}\end{aligned}}$$

This is the FDE for diffusion PDE using Crank-Nicolson method. Both temporal and spatial derivatives are approximated by second-order terms.

$\Rightarrow$ Another example of parabolic partial differential equation is Convection-Diffusion Equation (or Advection-Dispersion Equation).

$$\boxed{\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \alpha\frac{\partial^2 \phi}{\partial x^2}}$$

where $u \rightarrow$ convection velocity
$\alpha \rightarrow$ Diffusion coeff.

You may use the FTCS, or BTCS, or Crank-Nicolson schemes to solve the Convection - Diffusion equation.

⇒ Tomorrow we will discuss on Hyperbolic partial differential equations.