Lecture 4: Gauss Elimination Method

In the last class we discussed on:

* Matrix properties in brief
* Row operations on a matrix
* Determinants
* Cramer's rule, etc.

Yesterday while discussing about matrix properties on products

Say $[A][B] = [C]$

We had mistakingly said that

$$[A] = [B]^{-1}[C]$$

This is not true

As per matrix equations properties you need to post multiply the quantity with R.H.S of the equal sign.

$$\therefore [A] = [C][B]^{-1}$$

You can verify this for any simple matrix system

e.g. If $[A] = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$; $[B] = \begin{bmatrix} 2 & 5 \\ 9 & 4 \end{bmatrix}$

Then $[A][B] = [C] = \begin{bmatrix} 38 & 21 \\ 69 & 43 \end{bmatrix}$

$[B]^{-1}[C] = \begin{bmatrix} 5.22 & 3.54 \\ 5.51 & 2.78 \end{bmatrix} \neq [A]$ However, $[C][B]^{-1}$ gives $\begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$.

Gauss Elimination Method

→ The process of elimination method, whichever were done earlier is actually Gauss elimination method.

→ We need to find the Pivoting element for each elimination using scaling. Recall last day's lecture and the scaling example.

$$\begin{bmatrix} 3 & 2 & 105 \\ 2 & -3 & 103 \\ 1 & 1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 104 \\ 98 \\ 3 \end{Bmatrix}$$

We have suggested that the computer or instrument can store only three significant digits. If we directly do the pivoting and elimination, round-off errors will appear in the results.

→ Therefore, before doing first elimination, the first column of matrix is selected $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

→ This is not scaled. It has to be scaled with the highest number in the corresponding row $\rightarrow \begin{bmatrix} 3/105 \\ 2/103 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0.029 \\ 0.019 \\ 0.333 \end{bmatrix}$

→ On scaling, it ~~was~~ is seen that the third row is having the highest element. Therefore it will be taken as pivot element.

→ Pivoting is done by interchanging First and Third Rows.

Pivoting → $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 2 & -3 & 103 & 98 \\ 3 & 2 & 105 & 104 \end{array}\right]$ $\left.\begin{aligned} R_2 &= R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1 = R_2 - 2R_1 \\ R_3 &= R_3 - \left(\frac{a_{31}}{a_{11}}\right) R_1 = R_2 - 3R_1 \end{aligned}\right\}$ First Elimination

$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & -5 & 97 & 92 \\ 0 & -1 & 96 & 95 \end{array}\right]$ $R_3 = R_3 - \left(\frac{a_{32}}{a_{22}}\right) R_2 = R_3 - 0.2 R_2$

$$\left[\begin{array}{ccc|c} 1.0 & 1.0 & 3.0 & 3.0 \\ 0.0 & -5.0 & 97.0 & 92.0 \\ 0.0 & 0.0 & 76.6 & 76.6 \end{array}\right] \Rightarrow x_3 = 1.0$$

and back-substitute for others.

→ For solving large systems of linear algebraic equations, we use the aid of computers.

→ For that we need to direct the computer to do the process like — Scaling, Pivoting, and Elimination.

- ~~The~~ Gauss Elimination Process

$$\underset{n\times n}{[A]}\ \underset{n\times 1}{\{X\}} = \underset{n\times 1}{\{b\}}$$

If $[A]\{X\} = \{b\}$

Then we can write the same relation as

$$[U]\{X\} = \{Y\}$$

through systematic elimination

* Recall you can factorise $[A]$ as product of two matrices or you can do scaling, pivoting, and elimination to convert $[A] \rightarrow [U]$.

* The diagonal elements of each rows are the pivotal elements. (Ensure they are non-zero).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}$$

Step 1

* To reduce the elements of first column to zero, except the pivot element
* The pivot element is a_{11}. If $a_{11} = 0$ do pivoting by finding $a_{r1} = \max_{2 \le i \le n} |a_{i1}|$
* Identify multiplying factor for each row.
* Multiplying factors are $l_{21} = \frac{a_{21}}{a_{11}}$, $l_{31} = \frac{a_{31}}{a_{11}}$, $\frac{a_{41}}{a_{11}}, \dots, \frac{a_{i1}}{a_{11}}, \dots, \frac{a_{n1}}{a_{11}}$.
* ~~To make~~ The first column is $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}$ gener $l_{i1} = \frac{a_{i1}}{a_{11}}$, $i = 2, 3, \dots, n$

We have to make $a_{21}, a_{31}, \dots a_{n1}$ as zeros

* For that multiply the first row by multiplying factor and deduct it from the corresponding row

→ Note that corresponding changes occur in the R.H.S vector b.

i.e. $a_{ij}^{(1)} = a_{ij} - a_{1j}\, l_{i1}$

* ~~Doing~~ $b_i^{(1)} = b_i - l_{i1} b_1$

$$i = 2, 3, 4, \dots, n$$
$$j = 1, 2, 3, 4, \dots, n$$

* After first step

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2j}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \cdots & a_{3j}^{(1)} & \cdots & a_{3n}^{(1)} \\ \vdots & & & & & & \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nj}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{Bmatrix}$$

Step 2

* A similar procedure is adopted
* $a_{22}^{(1)}$ is selected as pivot element.

  If $a_{22}^{(1)} = 0$, then partial pivoting done among row 'r' and row 2. $a_{r2}^{(1)} = \max\limits_{3 \le i \le n} |a_{i2}^{(1)}|$

* All the elements below the pivotal element in column 2 is to be made zero.
* Develop multiplying factor for each row

$$l_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}}, \quad l_{42} = \frac{a_{42}^{(1)}}{a_{22}^{(1)}}, \quad \dots, \quad l_{n2} = \frac{a_{n2}^{(1)}}{a_{22}^{(1)}}$$

In general $l_{i2} = \dfrac{a_{i2}^{(1)}}{a_{22}^{(1)}}$; $i = 3, 4, 5, \dots, n$

* Eliminate all elements below $a_{22}^{(1)}$ in col. 2 as zero by:

$$\left.\begin{aligned} a_{ij}^{(2)} &= a_{ij}^{(1)} - l_{i2}\, a_{2j}^{(1)} \\ b_i^{(2)} &= b_i^{(1)} - l_{i2}\, b_2^{(1)} \end{aligned}\right\} \quad \begin{aligned} i &= 3, 4, 5, \dots, n \\ j &= 2, 3, 4, 5, \dots, n \end{aligned}$$

* So at the end of II^{nd} step:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2j}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & a_{34}^{(2)} \dots & a_{3j}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & 0 & a_{43}^{(2)} & a_{44}^{(2)} \dots & a_{4j}^{(2)} & \dots & a_{4n}^{(2)} \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & a_{n3}^{(2)} & a_{n4}^{(2)} \dots & a_{nj}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ b_4^{(2)} \\ \vdots \\ b_n^{(2)} \end{Bmatrix}$$

In general for k^{th} step

$$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - l_{ik}\, a_{kj}^{(k-1)} \\ b_i^{(k)} &= b_i^{(k-1)} - l_{ik}\, b_k^{(k-1)} \end{aligned}\right\}$$

$$i = k+1, k+2, \dots, n$$
$$j = k, k+1, k+2, \dots, n$$

$$k = 1, 2, 3, \dots, n-1$$

You can visualise how the coefft matrix will be at the end of k^{th} step.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & & a_{1k} & a_{1\,k+1} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & & a_{2k}^{(1)} & a_{2\,k+1}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & & a_{3k}^{(2)} & a_{3\,k+1}^{(2)} & \cdots & a_{3n}^{(2)} \\ 0 & 0 & 0 & a_{44}^{(3)} & \cdots & a_{4k}^{(3)} & a_{4\,k+1}^{(3)} & \cdots & a_{4n}^{(3)} \\ \vdots & \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & & & a_{kk}^{(k-1)} & a_{k\,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \\ \vdots & \vdots & \vdots & & & 0 & a_{k+1\,k+1}^{(k)} & \cdots & a_{k+1\,n}^{(k)} \\ \vdots & \vdots & \vdots & & & 0 & a_{k+2\,k+1}^{(k)} & \cdots & a_{k+2\,n}^{(k)} \\ \vdots & \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & - & - & 0 & a_{n\,k+1}^{(k)} & \cdots & a_{nn}^{(k)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ b_4^{(3)} \\ \vdots \\ b_k^{(k-1)} \\ b_{k+1}^{(k)} \\ b_{k+2}^{(k)} \\ \vdots \\ b_n^{(k)} \end{Bmatrix}$$

⇒ This is the systematic process of elimination to obtain Upper Triangular Matrix

* No. of steps required for a matrix [A] to be converted to upper triangular matrix [U] is $(n-1)$.

~~Final~~ Substitution Process

$$[U]\{x\} = \{y\}$$

where $\{y\}^T = \{ b_1 \quad b_2^{(1)} \quad b_3^{(2)} \quad \cdots \quad b_n^{(n-1)} \}$

Then $$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_{n-1} = \frac{b_{n-1}^{(n-2)} - a_{n-1,n}^{(n-2)}\, x_n}{a_{n-1,\,n-1}^{(n-2)}}$$

$$x_{n-2} = \frac{b_{n-2}^{(n-3)} - a_{n-2,n-1}^{(n-3)} x_{n-1} - a_{n-2\,n}^{(n-3)} x_n}{a_{n-2,n-2}^{(n-3)}}$$

$\vdots$

$$x_i = \frac{b_i^{(i-1)} - a_{i,i+1}^{(i-1)} x_{i+1} - a_{i,i+2}^{(i-1)} x_{i+2} - \cdots - a_{i,n}^{(i-1)} x_n}{a_{i,i}^{(i-1)}}$$

or
$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{i,i}^{(i-1)}} \quad ; \quad i=(n-1),(n-2),\ldots,2,1$$

This process is called Back-substitution.

Computations Required

Number of computations required to form upper triangular matrix from $[A]$

→ At each step k

1) $(n-k)$ computations to evaluate l_{ik}
2) $(n-k)(n-k+1)$ computations to evaluate a_{ij}
3) $(n-k)$ computations to evalute $b_i^{(k)}$