Lecture 39: Elliptic Partial Differential Equations (03-Nov-2012)

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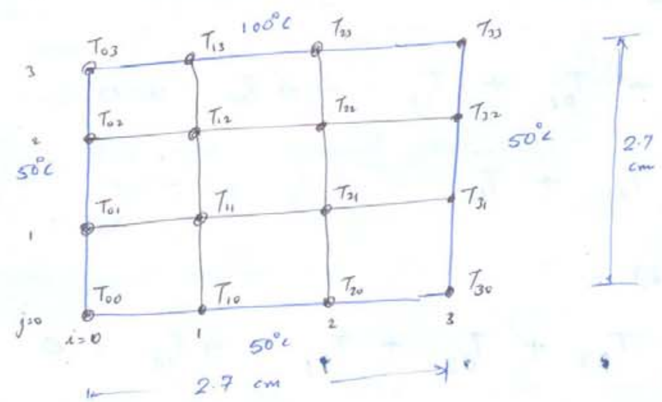
ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

We have seen how to formulate using finite-difference method the elliptic PDE's to algebraic FDE.

Let us demonstrate the method for a heat transfer problem in steady state. The problem domain is a steel plate with negligible thickness and 2.7 cm × 2.7 cm in dimensions. On the top edge the temperature is fixed at 100°C. On the ~~left~~ bottom edge temperature is fixed at 50°C. On the left and right edges the temperature is fixed at 50°C. Solve to obtain heat distribution in the plate.



The governing equation is $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

$$T(x = 2.7\text{cm}, y) = 50°C, \quad T(x, y = 2.7\text{cm}) = 100°C$$

$$T(x = 0, y) = 50°C, \quad T(x, y = 0) = 50°C.$$

Let us consider $\Delta x = \Delta y = 0.9$ cm

We have the grid point distribution as shown in figure.

The temperatures at grid points T_{11}, T_{12}, T_{21} and T_{22} are unknown to you.

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} \qquad O\,\Delta x^2$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \qquad O(\Delta y^2)$$

The FDE at any grid point (i,j) is:

$$T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1} - 4T_{ij} = 0 \rightarrow (1)$$

∴ At grid-point $(1,1)$

$$T_{21} + T_{12} + T_{01} + T_{10} - 4T_{11} = 0$$

i.e. $T_{21} + T_{12} - 4T_{11} = -100 \rightarrow$ (i) $\quad \left(\because \begin{array}{l} T_{01} = 50 \\ T_{10} = 50 \end{array}\right)$

At grid point $(1,2)$:

$$T_{22} + T_{13} + T_{02} + T_{11} - 4T_{12} = 0$$

i.e. $T_{11} + T_{22} - 4T_{12} = -150 \rightarrow$ (ii) $\quad \left(\because \begin{array}{l} T_{13} = 100 \\ T_{02} = 50 \end{array}\right)$

At grid point $(2,1)$:

$$T_{31} + T_{22} + T_{11} + T_{20} - 4T_{21} = 0$$

i.e. $T_{11} + T_{22} - 4T_{21} = -100 \rightarrow$ (iii)

At grid point $(2,2)$

$$T_{32} + T_{23} + T_{12} + T_{21} - 4T_{22} = 0$$

i.e. $T_{12} + T_{21} - 4T_{22} = -150 \rightarrow$ (iv)

These four equations

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{Bmatrix} = \begin{Bmatrix} -100 \\ -150 \\ -100 \\ -150 \end{Bmatrix}$$

On solving this system you will get the values of temperature at various location in the steel plate.

⇒ For elliptic PDE's there are three important properties of finite-difference equations

* Consistency
* Order
* Convergence

→ A FDE is consistent with a PDE – if the difference between FDE and PDE vanishes as grid sizes tends to zero.

→ The order of a FD - approximations of a PDE is the rate at which the error of the FD solution approaches zero as the grid size approach zero.

→ A FDM is convergent – if the solution of FDE approaches exact solution of PDE as $(\Delta x$ or $\Delta y \to 0)$.

⇒ The FDE of the PDE when applied on each grid-point yields a system of equations.

* You may also adopt iterative methods to solve the system.
  → Jacobi iteration
  → Gauss- Siedel
  → Successive- Over Relaxation

Gauss - Siedel Method

In the Laplace equation, the FDE at any grid point (i, j) was:

$$f_{i+1,j} + \beta^2 f_{i,j+1} + f_{i-1,j} + \beta^2 f_{i,j-1} - 2(1+\beta^2) f_{i,j} = 0$$

As per iterative technique :

$$\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \Delta\phi_{i,j}^{(k+1)}$$

where k is the iteration number.

You can correlate $\Delta\phi_{i,j}^{(k+1)}$ from the FDE as below

$$\Delta\phi_{i,j}^{(k+1)} = \frac{\phi_{i+1,j}^{(k)} + \beta^2\phi_{i,j+1}^{(k)} + \phi_{i-1,j}^{(k+1)} + \beta^2\phi_{i,j-1}^{(k+1)} - 2(1+\beta^2)\phi_{i,j}^{(k)}}{2(1+\beta^2)}$$

→ You start with some initial guess of $\phi_{i,j}$

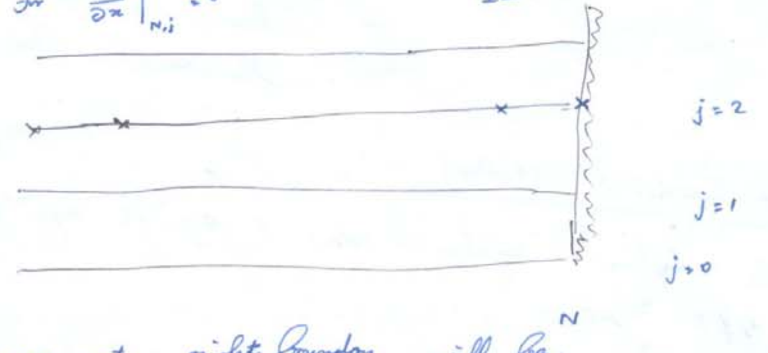
To <u>Solve Laplace Equation for Derivative B.C.'s</u>

If you have derivatives on boundary conditions

e.g. $\left.\frac{\partial T}{\partial x}\right|_{N,j} = 0$ or $\left.\frac{\partial T}{\partial y}\right|_{i,M} = 0$ etc.

You may adopt the same approach taught for BV-ODE and employ them here to get the solution.

e.g.: for $\left.\frac{\partial T}{\partial x}\right|_{N,j} = 0 \Rightarrow \frac{T_{N+1} - T_{N-1}}{2\Delta x} = 0$ or $T_{N+1} = T_{N-1}$



The FDE at right boundary will be:

$$\phi_{N+1,j} + \beta^2\phi_{N,j+1} + \phi_{N-1,j} + \beta^2\phi_{N,j-1} - 2(1+\beta^2)\phi_{N,j} = 0$$

On.

$$2\,\theta_{N-1,j} + \beta^2\left(\theta_{N,j+1} + \theta_{N,j-1}\right) - 2(1+\beta^2)\,\theta_{N,j} = 0$$

Finite-Difference Method for Poisson Equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = F(x,y)$$

∴ At any general grid-point (i,j)

$$\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{\Delta x^2} + \frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\Delta y^2} = F_{ij}$$

or Defining $\beta = \dfrac{\Delta x}{\Delta y}$

$$\theta_{i+1,j} + \beta^2 \theta_{i,j+1} + \theta_{i-1,j} + \beta^2 \theta_{i,j-1} - 2(1+\beta^2)\,\theta_{i,j} = \Delta x^2 F_{ij} \rightarrow (1)$$

You can use this FDE at each grid point to obtain the system of equations.

Non-rectangular Domains

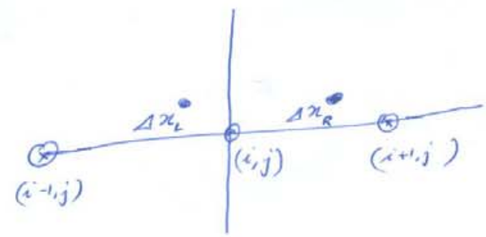
If you have observed, our finite-difference formulations were based on rectangular domains.

* In non-rectangular domains, the edges of rectangular grid cell may not coincide with the domain boundary.

You may have to then

- → Approximate the physical boundary
- → Use some other co-ordinate system
- → Use non-uniform finite-difference approximations
- → Or transform the non-rectangular space to rectangular.

Non-uniform finite difference approx.



Let us go back to Taylor series. Using the information at grid point (i,j)

$$f_{i+1,j} = f_{i,j} + \left.\frac{\partial f}{\partial x}\right|_{i,j} \Delta x_R + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)} (\Delta x_R)^2 + \frac{1}{6}\left.\frac{\partial^3 f}{\partial x^3}\right|_{(i,j)} (\Delta x_R)^3 + \cdots \rightarrow (1)$$

$$f_{i-1,j} = f_{i,j} + \left.\frac{\partial f}{\partial x}\right|_{i,j} (-\Delta x_L) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)} (\Delta x_L)^2 + \frac{1}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{(i,j)} (-\Delta x_L)^3 + \cdots \rightarrow (2)$$

Eq. (1) $\times \Delta x_L$ + Eqn (2) $\times \Delta x_R$, we will get

$$\Delta x_L f_{i+1,j} + \Delta x_R f_{i-1,j} = (\Delta x_L + \Delta x_R) f_{i,j} + \frac{1}{2}\left(\Delta x_L \Delta x_R^2 + \Delta x_R \Delta x_L^2\right)\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j}$$

$$+ \cdots$$

Truncating $\frac{\partial^3 f}{\partial x^3}$ onwards with $O(\Delta x^3)$ approximation.

$$\therefore \left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)} = \frac{2\Delta x_R f_{i-1,j} - 2(\Delta x_L + \Delta x_R) f_{i,j} + 2\Delta x_L f_{i+1,j}}{\left(\Delta x_L \Delta x_R^2 + \Delta x_R \Delta x_L^2\right)}$$