Lecture 38: Partial Differential Equations (02-Oct-2012)

LECTURE 38
02-NOV-2012

BV-ODE's

Yesterday you have seen the applications of Dirichlet and Neumann boundary conditions to the BV-ODE's.

Q: How will you now apply mixed boundary conditions?

S: A mixed boundary condition for a one-dimensional BV-ODE may look like this at (say R.H.S. boundary)

$$A\, y(x_2) + B \left.\frac{dy}{dx}\right|_{x_2} = C \quad \rightarrow (1)$$

For the general linear second-order BV-ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y = F(x)$$

The FDE at any arbitrary node i is:

$$\left(1 - P_i \frac{\Delta x}{2}\right) y_{i-1} + \left(-2 + \Delta x^2 Q_i\right) y_i + \left(1 + \frac{\Delta x}{2} P_i\right) y_{i+1} = \Delta x^2 F_i \quad \rightarrow (2)$$

At the right boundary node N

$$A\, y_N + B \left.\frac{dy}{dx}\right|_N = C$$

i.e. $$A\, y_N + B\, \frac{y_{N+1} - y_{N-1}}{2\Delta x} = C$$

or $$y_{N+1} = y_{N-1} - 2\frac{A}{B}\Delta x\, y_N + 2\frac{C}{B}\Delta x \quad \rightarrow (3)$$

When the FDE (2) is applied at node N, substitute the expression (3) for y_{N+1}.

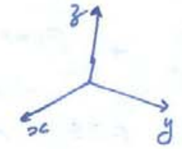
PARTIAL DIFFERENTIAL EQUATIONS

Most of the real physical processes in engineering and science are described ~~as~~ using partial differential equations. We can see how the finite-difference method is used to solve such PDE's.

Q: What is a partial differential equation?

A PDE → relates a dependent variable say C with respect to the independent variables x, y, z, and/or t.

$$\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} = 0$$



(Laplace Equation)

$$\frac{\partial C}{\partial t} = \alpha \frac{\partial^2 C}{\partial x^2} \quad ; \quad \text{(Diffusion equation)}$$

$$\frac{\partial^2 C}{\partial t^2} = m \frac{\partial^2 C}{\partial x^2} \quad \text{(Wave equation).}$$

⇒ The solution of PDE will be $C(x, y, z)$ or $C(x, t)$, $C(y, t)$, etc. in the suggested spatial and temporal domain.

⇒ Recall your mathematics courses:

Laplace equation : $\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} = 0$

$$\nabla^2 C = 0$$

$$\frac{\partial C}{\partial t} = \alpha \nabla^2 C \quad \text{Diffusion equation}$$

⇒ Laplace equation is homogeneous.

$$\nabla^2 c = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} = F(x, y, z)$$

This is Poisson's equation.

In PDE's you may find the dependent variable varying both w.r.t space and time.

Classification of PDE's

If there is a dependent variable f that may vary w.r.t space and/or time.

→ The general quasi-linear second-order non-homogeneous PDE in two independent variables is:

$$A \frac{\partial^2 f}{\partial x^2} + B \frac{\partial^2 f}{\partial x \partial y} + C \frac{\partial^2 f}{\partial y^2} + D \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial y} + F f = G$$

The discriminant of the coefficients is:

$B^2 - 4AC$

If $B^2 - 4AC$ ⇒ Negative Elliptic PDE
 eg: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

If $B^2 - 4AC$ ⇒ Zero Parabolic PDE
 eg: $\frac{\partial f}{\partial t} = \alpha \nabla^2 f$

If $B^2 - 4AC$ = Positive Hyperbolic PDE
 eg: $\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$

PDE's are usually used to describe problems that are:

→ Equilibrium problems eg: $\nabla^2 T = 0$

→ Propagation problems eg: $\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$

 " $\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 P$

→ Eigen problems

(Eigen problems → solution exists only for special values of a parameter of the problem).

As seen for ODE

→ You may have to provide IC for IV-ODE

→ " " " " " B.C. for BV-ODE

In PDE's you may have to apply

→ IC's and B.C's to the same problem.

→ You can have - Dirichlet, Neumann, and Mixed Boundary conditions.

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Elliptic PDE's are usually described for steady-state equilibrium problems.

Second-order quasi-linear PDE's

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{(Laplace Eqn.)}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = F(x,y) \quad \text{(Poisson Eqn.)}$$

⇒ We will employ Finite - Difference Method to solve PDE's.

⇒ Elliptic PDE's are closed domain problems.

⇒ The same processes of

→ Discretizing

→ Approximating partial derivatives by finite-difference formulas

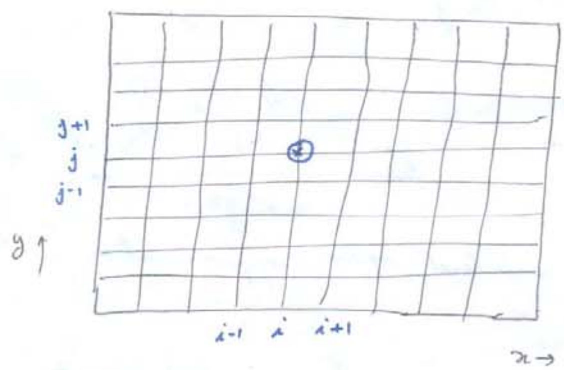
→ Substituting in PDE

→ Obtain FDE, etc.

are followed for solving PDE's.

As this is a closed domain (Elliptic PDE) problem.

Say the two-dimensional domain (x, y)



Finite-difference grid.

→ Discretize the spatial domain

→ Intersection of grid-lines called grid-points

→ The governing PDE is to be applied at each of these grid points.

i.e. $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)}$ or $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(i,j)}$, etc.

→ Approximate the derivatives by finite-diff formulas.

e.g. $$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(i,j)} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{\Delta x^2} \qquad O(\Delta x^2)$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(i,j)} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-i}}{\Delta y^2} \qquad O(\Delta y^2)$$

Finite - Difference Method for Laplace Equation

If there is a variable ζ that varies w.r.t x, y then the Laplace equation

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = 0 \longrightarrow (1)$$

If applied in the (x, y) domain

At any general ~~node~~ grid-point (i, j)

$$\frac{\zeta_{i+1,j} - 2\zeta_{i,j} + \zeta_{i-1,j}}{\Delta x^2} + \frac{\zeta_{i,j+1} - 2\zeta_{i,j} + \zeta_{i,j-1}}{\Delta y^2} = 0 \rightarrow (2)$$

(Recall this is second order centered space approximation for the derivative).

Equation (2) on rearranging and defining grid aspect ratio $\beta = \frac{\Delta x}{\Delta y}$, we have:

$$\zeta_{i+1,j} + \beta^2 \zeta_{i,j+1} + \zeta_{i-1,j} + \beta^2 \zeta_{i,j-1} - 2(1+\beta^2)\zeta_{i,j} = 0 \rightarrow (3)$$

Equation (3) is FDE of the Laplace equation.

If $\beta = 1$, then

$$\zeta_{i+1,j} + \zeta_{i,j+1} + \zeta_{i-1,j} + \zeta_{i,j-1} - 4\zeta_{i,j} = 0 \rightarrow (4)$$

⇒ There are informations of five grid-points involved in equation (4). It is a five-point scheme.

⇒ This FDE is applied at all the grid points in the domain. This will yield a system of algebraic equations.