Lecture 37: BV-ODE: Dirichlet and Neuman Boundary Conditions **(31-Oct-2012)**

LECTURE 37

31-OCT-2012

BOUNDARY-VALUE ODE's

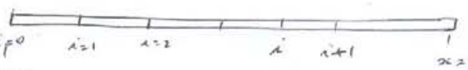
Yesterday, we started the discussion on Boundary-value ODE's. We used the boundary-value method or equilibrium method to solve BV-ODE's.

$\Rightarrow$ This is a finite-difference method.

Yesterday we asked in the quiz to obtain FDE for the BV-ODE

$$\frac{dc}{dx} = D\frac{d^2c}{dx^2}$$

Most of you have done correctly.



The One-Dimensional Domain

→ The domain is discretised to small grids nodes with unif. size Δx.

→ ∴ At any grid node 'i'

$$\left.\frac{dc}{dx}\right|_i = D_i \left.\frac{d^2c}{dx^2}\right|_i$$

$$\left.\frac{dc}{dx}\right|_i = \frac{C_{i+1} - C_{i-1}}{2\Delta x} + O(\Delta x^2)$$

$$\left.\frac{d^2c}{dx^2}\right|_i = \frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

$$\therefore \quad \frac{C_{i+1} - C_{i-1}}{2\Delta x} = D_i \frac{C_{i+1} - 2C_i + C_{i-1}}{\Delta x^2}$$

or $$C_{i+1} - C_{i-1} = \frac{2D_i}{\Delta x}\left[C_{i+1} - 2C_i + C_{i-1}\right]$$

or $$\left(1 + \frac{2D_i}{\Delta x}\right)C_{i-1} - \frac{4D_i}{\Delta x}C_i + \left(-1 + \frac{2D_i}{\Delta x}\right)C_{i+1} = 0$$

is the required FDE.

Example:

Solve the following heat transfer BV-ODE by equilibrium method.

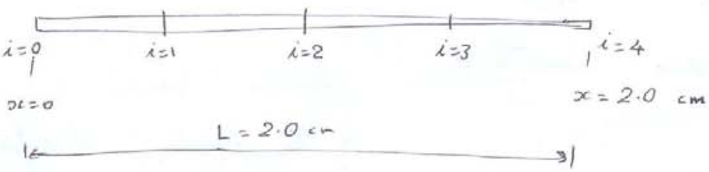
$$\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_a \quad ; \quad T(0.0) = 25^\circ C$$
$$T(2.0) = 100^\circ C$$

where x in cm, $\alpha^2 \rightarrow$ heat diffusivity = $16.0 \ cm^{-2}$

$T_a \rightarrow$ ambient temperature $= 20^\circ C$

Solution

This is a one dimensional problem.



The length of the rod is 2.0 cm.

The domain is discretised into small elements of $\Delta x = 0.5$ cm with grid nodes at $x = 0.50$, $x = 1.0$, and $x = 1.5$ cm

At any grid node 'i'.

$$\left.\frac{d^2T}{dx^2}\right|_i - 16\, T_i = -16 \times 20$$

i.e. $$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} - 16\, T_i = -320$$

$\Delta x = 0.50$ cm.

$$\therefore \quad T_{i+1} - 2T_i + T_{i-1} - 4\, T_i = -80$$

or $$T_{i-1} - 6T_i + T_{i+1} = -80$$

This is the FDE.

At node $i = 0$, $T_0 = T(0.0) = 25^\circ$ Already given

At node $i = 1$, $T_0 - 6T_1 + T_2 = -80$

i.e. $25 - 6T_1 + T_2 = -80$

or $-6T_1 + T_2 = -105$

$i = 2$, $T_1 - 6T_2 + T_3 = -80$

$i = 3$, $T_2 - 6T_3 + T_4 = -80$

i.e. $T_2 - 6T_3 + 100 = -80$

or $T_2 - 6T_3 = -180$

We have a system of equations with unknowns T_1, T_2, T_3.

$$\begin{bmatrix} -6 & 1 & 0 \\ 1 & -6 & 1 \\ 0 & 1 & -6 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} -105 \\ -80 \\ -180 \end{Bmatrix}$$

Solving this system, we get
$T_1 = 21.25^\circ c$
$T_2 = 22.5^\circ c$
$T_3 = 33.75^\circ c$

We may have to provide more grid nodes to get a more accurate picture of temperature distribution for this problem.

Q: How will you solve BV-ODE if you have derivatives as boundary conditions? (i.e. Neuman Conditions).

You can use the equilibrium method to solve such problems.

For any general second-order linear BV-ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = F(x) \; ;$$

$$y(x=0) = y_0$$

$$\left.\frac{dy}{dx}\right|_{x=L} = m$$

On discretising, approximating, and substituting – we get the finite-difference equation at any general node 'i'

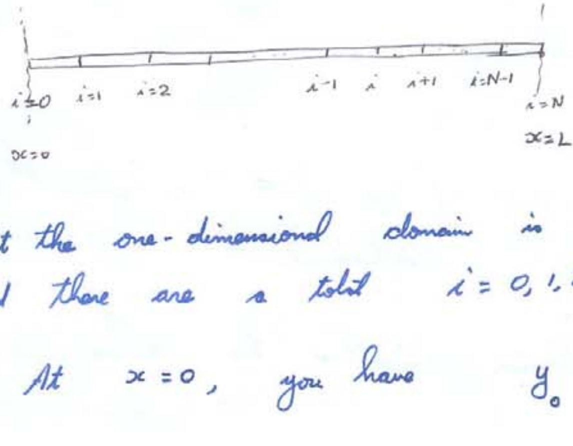
$$\left.\frac{d^2y}{dx^2}\right|_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \rightarrow O(\Delta x^2)$$

$$\left.\frac{dy}{dx}\right|_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \rightarrow O(\Delta x^2)$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + P_i\,\frac{y_{i+1} - y_{i-1}}{2\Delta x} + Q_i\,y_i = F_i$$

Rearranging :

$$\left(1 - P_i\frac{\Delta x}{2}\right)y_{i-1} + \left(-2 + \Delta x^2 Q_i\right)y_i + \left(1 + \frac{\Delta x}{2}P_i\right)y_{i+1} = \Delta x^2 F_i \quad \rightarrow (1)$$



Let the one-dimensional domain is discretised into small elements and there are a total $i = 0, 1, 2, 3, \ldots, N$ grid points.

$\Rightarrow$ At $x = 0$, you have $y_0 = y_0$

However, at $x = L$, you have $\left.\frac{dy}{dx}\right|_N = m$

How will you evaluate $\left.\frac{dy}{dx}\right|_N$ at the node N.

→ For that an imaginary node at same distance Δx from $i = N$ is described. This is designated as $i = N+1$.

$$\therefore \left.\frac{dy}{dx}\right|_N = \frac{y_{N+1} - y_{N-1}}{2\Delta x} + O(\Delta x^2)$$

As $\left.\frac{dy}{dx}\right|_N = m = \frac{y_{N+1} - y_{N-1}}{2\Delta x}$

$$\text{or} \quad y_{N+1} = y_{N-1} + 2m\Delta x \quad \rightarrow ②$$

Applying FDE ① at node $i = N$

$$\left(1 - P_N \frac{\Delta x}{2}\right) y_{N-1} + \left(-2 + \Delta x^2 Q_N\right) y_N + \left(1 + \frac{\Delta x}{2} P_N\right) y_{N+1} = \Delta x^2 F_N \quad \rightarrow ③$$

Substitute ② in ③ to remove y_{N+1} term

$$\therefore \left(1 - P_N \frac{\Delta x}{2}\right) y_{N-1} + \left(-2 + \Delta x^2 Q_N\right) y_N + \left(1 + \frac{\Delta x}{2} P_N\right)\left(y_{N-1} + 2m\Delta x\right) = \Delta x^2 F_N$$

i.e.

$$2 y_{N-1} + \left(-2 + \Delta x^2 Q_N\right) y_N = \Delta x^2 F_N - 2m\Delta x - m\Delta x^2 P_N \quad \rightarrow ④$$

Equation ④ becomes the finite-difference equation at node N (i.e. the boundary node) for Neumann condition.

Example

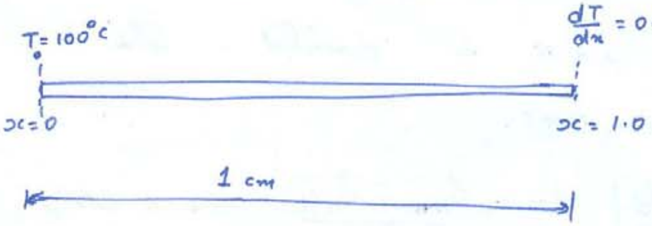
For the same heat transfer problem $\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_a$,

suppose if $T(x=0) = 100°c$ and $\left.\frac{dT}{dx}\right|_{x=1.0} = 0.0$

and $\alpha^2 = 16.0 \ cm^{-2}$, $T_a = 20°c$, Solve using $\Delta x = 0.25 cm$.

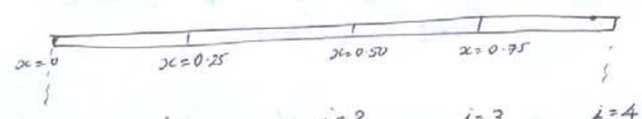
Solution.

Given



$$\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_a \longrightarrow (1)$$

As $\Delta x = 0.25$ cm



$$\left.\frac{d^2T}{dx^2}\right|_i = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

$$\therefore \frac{T_{i+1} - 2T_i + T_{i-1}}{(0.25)^2} - 16 T_i = -16 \times 20$$

$$\text{or} \quad T_{i-1} - 3T_i + T_{i+1} = -20 \longrightarrow (2)$$

At node $i = N = 4$ the boundary node

$$\cancel{2T_3 + (-2 - \alpha^2 \Delta x^2) T_4} = \left.\frac{dT}{dx}\right|_N = 0$$

$$\text{i.e} \quad \frac{T_5 - T_3}{2\Delta x} = 0 \quad \text{or} \quad T_5 = T_3$$

$\therefore$ Equation (2) at $i = N = 4$ becomes:

$$T_3 - 3T_4 + T_5 = -20$$

i.e. $\quad 2T_3 - 3T_4 = -20 \quad \longrightarrow (3)$

At Node $i = 0, \quad T_0 = 100$

$i = 1, \quad T_0 - 3T_1 + T_2 = -20$

i.e. $\quad -3T_1 + T_2 = -120$

$i = 2, \quad T_1 - 3T_2 + T_3 = -20$

$i = 3, \quad T_2 - 3T_3 + T_4 = -20$

$i = 4, \quad 2T_3 - 3T_4 = -20$

$\therefore$ The system is

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 2 & -3 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} -120 \\ -20 \\ -20 \\ -20 \end{Bmatrix}$$

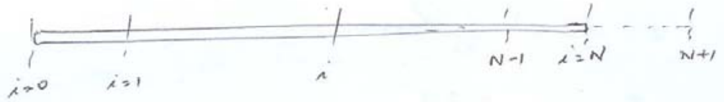
Solve to get T_1, T_2, T_3 & T_4.

Q: Sometimes mixed types of boundary conditions may also be applied in certain situations. How?

A mixed boundary condition, say at the R.H.S of the domain (i.e $x = x_2$), may look like this

$$A\, y(x_2) + B \left.\frac{dy}{dx}\right|_{x_2} = C \quad \rightarrow (1)$$

$\Rightarrow$ You can employ the equilibrium method here as well:



~~If you have~~ A

For a general linear second-order BV-ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

the FDE is:

$$\left(1 - P_i \frac{\Delta x}{2}\right) y_{i-1} + \left(-2 + \Delta x^2 Q_i\right) y_i + \left(1 + \frac{\Delta x}{2} P_i\right) y_{i+1} = \Delta x^2 F_i \rightarrow (2)$$

For a mixed boundary condition at right side

$$A\, y_N + B \left.\frac{dy}{dx}\right|_N = C$$

i.e. $$A\, y_N + B\, \frac{y_{N+1} - y_{N-1}}{2\Delta x} = C$$

or $$2A\,\Delta x\; y_N + B\, y_{N+1} - B\, y_{N-1} = 2\Delta x\, C$$

i.e. $$y_{N+1} = y_{N-1} - \frac{2A}{B}\Delta x\; y_N + \frac{2C}{B}\Delta x \rightarrow (3)$$

Substitute this equation (3) in equation (2) while applying FDE at node N (the right boundary).