Lecture 36: Multi-point methods

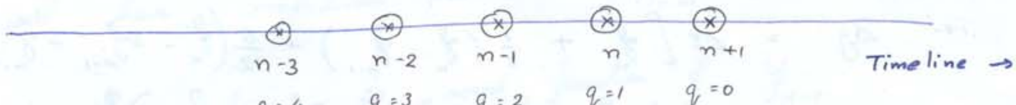
Boundary-Value Differential Equations (30-Oct-2012)

LECTURE 36

30-OCT-2012

Multi-Point Methods for IV-ODE

Yesterday we started the discussion on multi-point finite-difference methods to solve IV-ODE.



When $q=1$, $$\int_{y_n}^{y_{n+1}} dy = \Delta y = \int_{t_n}^{t_{n+1}} P_k(t)\, dt$$

Gives Adams - finite difference equations.

In the process, we discussed on Adams-Bashforth explicit and Adams-Moulton implicit 4^{th} order finite-difference equations, which we can again discuss as below.

⇒ The 4^{th} order Adams-Bashforth explicit equation was developed using a third-degree Newton's backward difference polynomial, ~~at the~~ with its base point at time t_n.

$$P_3(s) = f_n + s\nabla f_n + \frac{s(s+1)}{2}\nabla^2 f_n + \frac{s(s+1)(s+2)}{6}\nabla^3 f_n + O(\Delta t^4)$$

t	f	∇f	$\nabla^2 f$	$\nabla^3 f$
t_{n-3}	f_{n-3}			
		$f_{n-2}-f_{n-3}$		
t_{n-2}	f_{n-2}		$f_{n-1}-2f_{n-2}+f_{n-3}$	
		$f_{n-1}-f_{n-2}$		$f_n-3f_{n-1}+3f_{n-2}-f_{n-3}$
t_{n-1}	f_{n-1}		$f_n-2f_{n-1}+f_{n-2}$	
		f_n-f_{n-1}		$f_{n+1}-3f_n+3f_{n-1}-f_{n-2}$
t_n	f_n		$f_{n+1}-2f_n+f_{n-1}$	
		$f_{n+1}-f_n$		
t_{n+1}	f_{n+1}			

$$I = \int_{y_n}^{y_{n+1}} dy = \Delta y = \int_{t_n}^{t_{n+1}} \left[P_3(t)\right]_n dt$$

i.e. As $s = \dfrac{t - t_n}{\Delta t}$, $ds = \dfrac{dt}{\Delta t}$

$$\therefore \quad \Delta y = \Delta t \int_0^1 \left[f_n + s \nabla f_n + \frac{(s^2+s)}{2} \nabla^2 f_n + (s^3 + 3s^2 + 2s) \nabla^3 f_n + O(\Delta t^4) \right] ds$$

$$\text{i.e.} \quad \Delta y = \Delta t \left[f_n + \tfrac{1}{2}(f_n - f_{n-1}) + \tfrac{5}{12}(f_n - 2f_{n-1} + f_{n-2}) + \tfrac{9}{24}(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}) + O(\Delta t^4) \right]$$

$$\text{i.e.} \quad \Delta y = \frac{\Delta t}{24} \left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3} \right] + O(\Delta t^5)$$

$\Rightarrow$ In a similar way you can develop 4th order Adams-Moulton implicit finite-difference equation.

$$\Delta y = \frac{\Delta t}{24} \left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2} \right]$$

$\Rightarrow$ The Adams-Bashforth – Moulton predictor-corrector approach to solve the IV-ODE is as given below:

$$\boxed{\begin{aligned} y^P_{n+1} &= y_n + \frac{\Delta t}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right] \\ y^C_{n+1} &= y_n + \frac{\Delta t}{24}\left[9 f^P_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right] \end{aligned}}$$

Non-linear Expressions In Implicit Finite-Difference Eqns.

You have seen that to solve:

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

$\Rightarrow$ You can use explicit and implicit schemes.

If $f(t, y)$ is non-linear in y, then in the implicit formulations

e.g. say: Euler's $y_{n+1} = y_n + \Delta t\, f(t_{n+1}, y_{n+1})$

will have non-linear term in y_{n+1}

i.e. y_{n+1} is some function of y_{n+1}

i.e. $y_{n+1} = G(y_{n+1})$

or $y_{n+1} - G(y_{n+1}) = 0 = F(y_{n+1})$

Recall Newton's method (iterative) and implement here to obtain y_{n+1}

i.e. $$y^{(k+1)}_{n+1} = y^{(k)}_{n+1} - \frac{F(y^{(k)}_{n+1})}{F'(y^{(k)}_{n+1})}$$

Example: Let us recall our same example

$$\frac{dT}{dt} = -\alpha\,(T^4 - T_a^{\,4}) \quad ; \quad T_0 = 2500\ K, \quad \alpha = 4\times10^{-12}, \quad T_a = 250\ K$$

$\Rightarrow$ This IV-ODE is non-linear in T.

The modified Euler's implicit scheme gives

$$T_{n+1} = T_n + \frac{\Delta t}{2}\left[f_n + f_{n+1} \right]$$

$$f = -4\times10^{-12}\,(T^4 - 250^4)$$

$\Rightarrow$ Using $\Delta t = 2.0$ seconds,

$$T_1 = T_0 + \frac{2}{2}\left[-4\times10^{-12}\left(2500^4 - 250^4 + T_1^{\,4} - 250^4\right)\right]$$

i.e. $T_1 = 2500 - 4\times10^{-12}\left(3.90546875\times10^{-13} + T_1^4\right)$

i.e. $T_1 = G(T_1)$

, where $G(T_1) = 2500 - 156.21875 - 4\times10^{-12}\,T_1^4$

and $F(T_1) = T_1 - 2343.78125 + 4\times10^{-12}\,T_1^4$

$F'(T_1) = 1 - 1.6\times10^{-11}\,T_1^3$.

∴ Start with initial guess for $T_1 = 2500$ same as T_0.

$$T_1^{(1)} = T_1^{(0)} - \frac{F(T_1^{(0)})}{F'(T_1^{(0)})}$$

$F(T_1^{(0)}) = 312.46875$

$F'(T_1^{(0)}) =$

$$= 2500 - \frac{312.46875}{0.75}$$

$$= 2083.375 \text{ K}$$

$F(T_1^{(1)}) = -185.048$

$F'(T_1^{(1)}) = 0.855315$

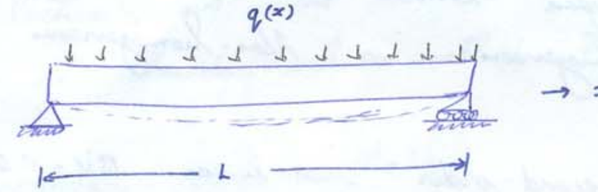
$$T_1^{(2)} = T_1^{(1)} - \frac{F(T_1^{(1)})}{F'(T_1^{(1)})} = 2083.375 - \frac{-185.048}{0.855315}$$

$$= 2299.725602$$

Similarly continue the iterations till convergence for each time step.

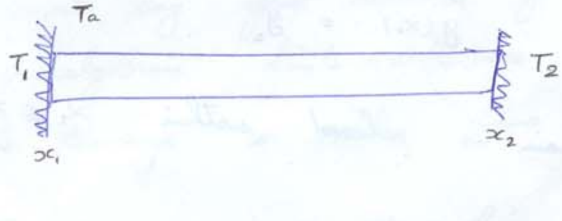
BOUNDARY - VALUE DIFFERENTIAL EQUATIONS

As a civil engineer, most of you might have studied the deflection of a beam



$$E\,I(x)\,\frac{d^4y}{dx^4} = q(x) \quad ; \quad \begin{array}{l} y(0) = 0 \\ y(L) = 0 \\ y''(0) = 0 \\ y''(L) = 0 \end{array}$$

⇒ The heat diffusion in a steel rod



$$\frac{d^2T}{dx^2} - \alpha^2 T = -\alpha^2 T_a \quad ; \quad \begin{array}{l} T(x_1) = T_1 \\ T(x_2) = T_2 \end{array}$$

⇒ These are boundary - value differential equations.

⇒ Boundary - value ODE's govern equilibrium problems.

→ Applied for closed domains.

In engineering and science, the boundary-value problems
→ can be of single dependent variable
→ Coupled systems of several dependent variables
→ Linear and Non-linear
→ Homogeneous or Non-homogeneous

Consider the second-order non-linear BV-ODE

$$\frac{d^2y}{dx^2} + P(x,y)\frac{dy}{dx} + Q(x,y)y = F(x) \;;$$

$$y(x_1) = y_1$$
$$y(x_2) = y_2$$

Second-order linear BV-ODE

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x) \;;$$

$$y(x_1) = y_1$$
$$y(x_2) = y_2$$

→ The solution domain is closed within $x_1 \le x \le x_2$.

Q: What are the types of boundary conditions for BV-ODE ?

S: You need to specify certain conditions at the boundaries of the closed domain
(i) The value of the dependent variable $y(x)$ specified at boundaries (Dirichlet)
(ii) The derivative of the dependent variable specified (Neuman BC)

(iii) Combination of $y(x)$ and $\frac{dy}{dx}$ specified at the Boundaries (Mixed BC's).

In our lecture, we will be discussing on the equilibrium or Boundary-value method to solve the BV-ODE.

The Boundary - Value Method

The finite-difference approach here consists of

→ Discretise the continuous (spatial) domain into discrete finite-difference grid points

→ Approximate exact derivatives in the BV-ODE by finite-difference formulas

→ Substitute these formulas in the original BV-ODE at any ~~grid~~ arbitrary grid-point to obtain algebraic finite-difference equation.

→ Rearrange or solve these finite-difference equations.

Consider the second-order linear BV-ODE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = F(x)$$

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2$$

Discretising the continuous domain

* As this is one-dimensional, the domain in x-direction

is discretized into definite node points (In the figure 8 node points).

* To substitute derivatives by finite-difference formulas:

At any general node point "i", the ODE is:

$$\left.\frac{d^2y}{dx^2}\right|_i + P(x_i)\left.\frac{dy}{dx}\right|_i + Q(x_i)\,y_i = F(x_i)$$

$$\left.\frac{d^2y}{dx^2}\right|_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

$$\left.\frac{dy}{dx}\right|_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x} + O(\Delta x^2)$$

* Substitute in the ODE:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} + P_i\,\frac{y_{i+1} - y_{i-1}}{2\Delta x} + Q_i\,y_i = F_i$$

* Rearrange the terms:

$$y_{i-1}\left(\frac{1}{\Delta x^2} - \frac{P_i}{2\Delta x}\right) + y_i\left(-\frac{2}{\Delta x^2} + Q_i\right) + \left(\frac{1}{\Delta x^2} + \frac{P_i}{2\Delta x}\right)y_{i+1} = F_i$$

or

$$\boxed{\left(1 - \frac{\Delta x}{2}P_i\right)y_{i-1} + \left(-2 + \Delta x^2 Q_i\right)y_i + \left(1 + \frac{\Delta x}{2}P_i\right)y_{i+1} = F_i} \quad \rightarrow (1)$$

* Apply this finite-difference equation (1) in each grid point (or nodes).