Lecture 35: Runge-Kutta Method; Multi-point methods

(29-Oct-2012)

IV-ODE's

LECTURE - 35
29-OCT-2012

Last time we were discussing on the Runge-Kutta methods to solve the Initial Value Ordinary Differential Equations.

For any IV-ODE

$$\frac{dy}{dt} = +f(t,y) \quad ; \quad y(t_0) = y_0$$

→ The FDM suggested

$$y_{n+1} = y_n + \Delta t\, f_n \quad \text{(Explicit)}$$

or

$$y_{n+1} = y_n + \Delta t\, f_{n+1} \quad \text{(Implicit)}, \quad \text{etc.}$$

→ This can be correlated as:

$$y_{n+1} = y_n + \underbrace{\Delta y}_{\text{Some change in } y}$$

→ This Δy was defined as:

$$\Delta y = \sum_k C_k\, \Delta y_k$$

where

$$\Delta y_k = \Delta t\, \underbrace{f(t,y)}$$

$f(t,y)$ evaluated at some point in the range $t_n \leq t \leq t_{n+1}$

k → the number of evaluations to be summed so as to decide the order of R-K method.

Last week we discussed the second-order R-K method.

$$y_{n+1} = y_n + c_1 \Delta y_1 + c_2 \Delta y_2$$

c_1 and c_2 are unknown at present.

$$\Delta y_1 = \Delta t \, f_n$$

$$\Delta y_2 = \Delta t \, f(t_n + \alpha \Delta t, \; y_n + \beta \Delta y_1)$$

Again α and β are also unknowns.

(Also see that for Δy_2 we are evaluating f at an instant t between t_n & t_{n+1})

Q: How are you going to find c_1, c_2, α, and β for the second-order R-K method ?

The second order R-K method suggests

$$y_{n+1} = y_n + c_1 \Delta t f_n + c_2 \Delta t f(t_n + \alpha \Delta t, \; y_n + \beta \Delta y_1) \rightarrow (1)$$

Keep the time grid point 'n' as base point and using Taylor's series

$$f(t_n + \alpha \Delta t, \; y_n + \beta \Delta y_1) = f_n + \alpha \Delta t \left.\frac{\partial f}{\partial t}\right|_n + \beta \Delta y_1 \left.\frac{\partial f}{\partial y}\right|_n + \cdots \rightarrow (2)$$

Substituting this expression in (1) :

$$y_{n+1} = y_n + c_1 \Delta t f_n + c_2 \Delta t \left[f_n + \alpha \Delta t \left.\frac{\partial f}{\partial t}\right|_n + \beta \Delta t f_n \left.\frac{\partial f}{\partial y}\right|_n + \cdots \right]$$

$$= y_n + (c_1 + c_2) \Delta t f_n + c_2 \Delta t^2 \left(\alpha \left.\frac{\partial f}{\partial t}\right|_n + \beta f_n \left.\frac{\partial f}{\partial y}\right|_n + \cdots \right) \rightarrow (3)$$

The unknown ~~quantities~~ parameters being c_1, c_2, α, and β

Using Taylor's series for $\dot{y}$ with base-point as y_n

$$y_{n+1} = y_n + \Delta t \left.\frac{dy}{dt}\right|_n + \frac{\Delta t^2}{2} \left.\frac{d^2y}{dt^2}\right|_n + \cdots\cdots$$

As $$\left.\frac{dy}{dt}\right|_n = f(t_n, y_n) = f_n$$

$$\therefore \left.\frac{d^2y}{dt^2}\right|_n = \left.\frac{d}{dt}\left(\frac{dy}{dt}\right)\right|_n = \left.\frac{df}{dt}\right|_n = \left.\frac{\partial f}{\partial t}\right|_n + \left.\frac{\partial f}{\partial y}\right|_n \left.\frac{dy}{dt}\right|_n$$

$$= \left.\frac{\partial f}{\partial t}\right|_n + f_n \left.\frac{\partial f}{\partial y}\right|_n$$

$$\therefore y_{n+1} = y_n + \Delta t f_n + \frac{\Delta t^2}{2}\left[\left.\frac{\partial f}{\partial t}\right|_n + f_n \left.\frac{\partial f}{\partial y}\right|_n\right] + \cdots \rightarrow (4)$$

Comparing equations (3) and (4) we get

$$c_1 + c_2 = 1 \rightarrow (a)$$
$$c_2 \alpha = \tfrac{1}{2} \rightarrow (b)$$
$$c_2 \beta = \tfrac{1}{2} \rightarrow (c)$$

You can have many possibilities of c_1, c_2, α, and β adhering to the relations (a), (b), and (c)

For example if $c_1 = \tfrac{1}{2}$,

Then $c_2 = \frac{1}{2}$, and $\alpha = 1$, $\beta = 1$

You will get $$\Delta y_2 = \Delta t \, f(t_n + \Delta t, \; y_n + \Delta y_1)$$
$$= \Delta t \, f_{n+1}$$

and $$y_{n+1} = y_n + \frac{1}{2}\Delta t f_n + \frac{1}{2}\Delta t f_{n+1}$$

(Same as Modified Euler's method).

4^{th} Order R-K Method

One can also formulate a 4^{th} order R-K method as was shown for 2^{nd} order R-K method

i.e. $y_{n+1} = y_n + c_1 \Delta y_1 + c_2 \Delta y_2 + c_3 \Delta y_3 + c_4 \Delta y_4$

One such 4^{th} order R-K method is:

$$y_{n+1} = y_n + \frac{1}{6}\left[\Delta y_1 + 2\Delta y_2 + 2\Delta y_3 + \Delta y_4\right]$$

where

$$\Delta y_1 = \Delta t\, f_n$$
$$\Delta y_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta y_1}{2}\right)$$
$$\Delta y_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta y_2}{2}\right)$$
$$\Delta y_4 = \Delta t\, f\left(t_n + \Delta t,\ y_n + \Delta y_3\right)$$

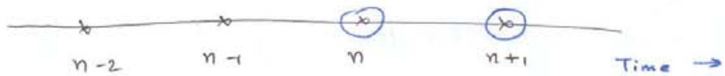
$\Rightarrow$ ~~The foll~~ One example on 4^{th} order R-K method is already uploaded in last lecture notes. You may go through them.

Multi - Point Methods

Recall that till now we were dealing with single-point methods to solve the IV- ODE

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

That is, if we have the information at time t_n then the solution at time t_{n+1} was obtained using only the information at time t_n.



If we use information from more than one previous instant, then the method will be a multi-point method.

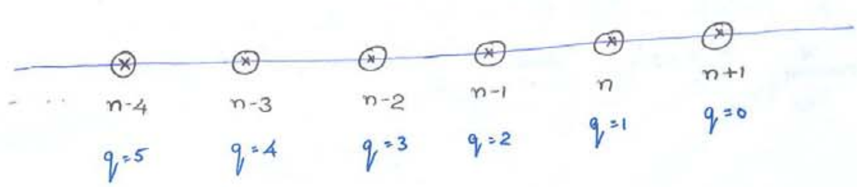
→ Again you can have

- Explicit
- Implicit

schemes for solving the IV-ODE using multi-points

↳ Multi-point methods are also called multi-step or multi-value methods.

Consider the timeline below:



For the IV- ODE

$$\frac{dy}{dt} = f(t, y)$$

We can approximate

$$dy = f(t, y(t))\, dt$$
$$= F(t)\, dt$$

$$\therefore \int dy = \int F(t)\, dt$$

Now if this $F(t)$ is approximated by Newton's backward difference polynomials, then

$$\int_{y_{(n+1-q)}}^{y_{n+1}} dy = \int_{t_{n+1-q}}^{t_{n+1}} [P_k(t)]\, dt$$

Now if the polynomial $P_k(t)$ is obtained with base point as time t_n, then the expression becomes explicit multi-step equation.

If $P_k(t)$ is obtained with base point at time t_{n+1}, then it becomes implicit multi-step formulation.

⇒ Suppose if we want change in y from time t_n to time t_{n+1}

i.e. $$\Delta y = \int_{y_n}^{y_{n+1}} dy = \int_{t_n}^{t_{n+1}} [P_k(t)]\, dt$$

i.e. $q = 1$ for such situation.

The resulting FDE's are Adams FDE.

* Explicit equations are called Adams-Bashforth FDE's
* Implicit equations are called Adams-Moulton FDE's.

As seen in single-point Euler method, you can have improvements by predictor-corrector methods.

Fourth-Order Adams-Bashforth-Moulton Method

We use a third degree polynomial to approximate $F(t)$

$$I = \int_{y_n}^{y_{n+1}} dy = \int_{t_n}^{t_{n+1}} F(t)\, dt = \int_{t_n}^{t_{n+1}} \left[P_3(t)\right]_n dt$$

Explicit expression

Recall Newton's backward difference polynomial based at time t_n.

$$P_3(s) = f_n + s \nabla f_n + \frac{s(s+1)}{2} \nabla^2 f_n + \frac{s(s+1)(s+2)}{6} \nabla^3 f_n + \frac{s(s+1)(s+2)(s+3)}{24} \Delta t^4 f^{(4)}(\xi, y(\xi))$$

where $s = \frac{t - t_n}{\Delta t}$ and $t_n \le \xi \le t_{n+1}$

$ds = \frac{dt}{\Delta t}$

Now At $t = t_n$; $s = 0$

$t = t_{n+1}$; $s = 1$

$$\therefore \underbrace{y_{n+1} - y_n}_{\int_{y_n}^{y_{n+1}} dy} = \Delta t \int_0^1 P_3(s) + \Delta t \int_0^1 \text{Error}(s)\, ds$$

i.e. $$y_{n+1} = y_n + \Delta t \int_0^1 \left[f_n + s \nabla f_n + \frac{s(s+1)}{2} \nabla^2 f_n + \frac{s(s+1)(s+2)}{6} \nabla^3 f_n \right] ds + \text{Error}.$$

i.e. $y_{n+1} = y_n + \Delta t . \left[f_n + \frac{1}{2}\nabla f_n + \frac{5}{12}\nabla^2 f_n + \frac{3}{8}\nabla^3 f_n \right] + \cdots$

t	f	∇f	$\nabla^2 f$	$\nabla^3 f$
t_{n-3}	f_{n-3}			
		$f_{n-2} - f_{n-3}$		
t_{n-2}	f_{n-2}		$f_{n-1} - 2f_{n-2} + f_{n-3}$	
		$f_{n-1} - f_{n-2}$		$f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}$
t_{n-1}	f_{n-1}		$f_n - 2f_{n-1} + f_{n-2}$	
		$f_n - f_{n-1}$		$f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2}$
t_n	f_n		$f_{n+1} - 2f_n + f_{n-1}$	
		$f_{n+1} - f_n$		
t_{n+1}	f_{n+1}			

$$\therefore y_{n+1} = y_n + \Delta t \left[f_n + \frac{1}{2}(f_n - f_{n-1}) + \frac{5}{12}(f_n - 2f_{n-1} + f_{n-2}) + \frac{3}{8}(f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3}) \right] + O(\Delta t^5)$$

i.e. $$\boxed{y_{n+1} = y_n + \frac{\Delta t}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3} \right]}$$

This is 4th Order Adams-Bashforth explicit FDE.

In a similar way you can get implicit FDE

$$\boxed{y_{n+1} = y_n + \frac{\Delta t}{24}\left[9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2} \right]}$$

This is Adams-Moulton 4th order FDE.

The predictor-corrector approach Adams-Bashforth-Moulton is:

$$\boxed{\begin{aligned} y^P_{n+1} &= y_n + \frac{\Delta t}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3} \right] \\ y^C_{n+1} &= y_n + \frac{\Delta t}{24}\left[9 f^P_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2} \right] \end{aligned}}$$