Lecture 34: Finite-Difference Method for IV-ODE

(19-Oct-2012)

LECTURE 34
19-10-2012

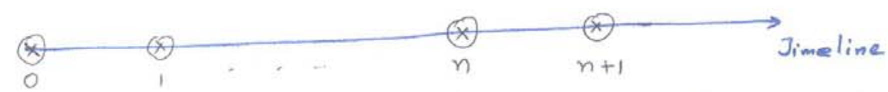
Finite-Difference Methods for Initial-Value ODE (Part-2)

Yesterday in the quiz, it was asked to you to show that implicit finite-difference Euler method is consistent.

$$\frac{dy}{dt} + \alpha y = F(t) \quad ; \quad y(t_0) = y_0 \quad \rightarrow (1)$$

(No one has given the answer correctly)

The solution is:



The IV-ODE $\frac{dy}{dt} + \alpha y = F(t) \quad ; \quad y(t_0) = y_0$

is representing some physical phenomenon.

Therefore, the phenomenon should be represented in L.H.S and R.H.S of the equation at same time.

e.g. Equation (1) at time step 'n' can be given as:

$$\left.\frac{dy}{dt}\right|_n + \alpha y_n = F_n \quad \rightarrow (2A)$$

Similarly at time step 'n+1', it can be given as:

$$\left.\frac{dy}{dt}\right|_{n+1} + \alpha y_{n+1} = F_{n+1} \quad \rightarrow (2B)$$

So the implicit finite difference ~~form~~ Euler equation for (1) will be

$$y_{n+1} = y_n + \Delta t \left[F_{n+1} - \alpha y_{n+1}\right] \rightarrow (3)$$

Keep the base point as n+1 and obtain Taylor's series for y_n

$$y_n = y_{n+1} + (-\Delta t)\left.\frac{dy}{dt}\right|_{n+1} + \frac{\Delta t^2}{2}\left.\frac{d^2y}{dt^2}\right|_{n+1} + \cdots$$

i.e. $$y_{n+1} = y_n + \Delta t\left.\frac{dy}{dt}\right|_{n+1} - \frac{\Delta t^2}{2}\left.\frac{d^2y}{dt^2}\right|_{n+1} + \cdots \quad \to 4$$

Comparing 3 and 4, ~~we get~~ and dividing by Δt we get:

$$F_{n+1} - \alpha y_{n+1} = \left.\frac{dy}{dt}\right|_{n+1} - \frac{\Delta t}{2}\left.\frac{d^2y}{dt^2}\right|_{n+1} \quad \to 5$$

As $\Delta t \to 0$, Equation 5 becomes

$$\left.\frac{dy}{dt}\right|_{n+1} + \alpha y_{n+1} = F_{n+1} \quad \text{same as } (2B)$$

$\therefore$ Implicit Euler method is consistent

Regarding Stability of Modified Euler Method

We have $\cancel{y^P_{n+1} = y_n + \frac{\Delta t}{2}\left(f_n + f_{n+1}\right)}$

$$y^P_{n+1} = y_n + \Delta t\, f_n$$
$$y^c_{n+1} = y_n + \frac{\Delta t}{2}\left[f_n + f^P_{n+1}\right]$$

For the IV-ODE $\quad \frac{dy}{dt} + \alpha y = 0$

$$y^p_{n+1} = y_n + \Delta t \cdot (-\alpha y_n)$$
$$= (1 - \alpha \Delta t)\, y_n$$

$$y^c_{n+1} = y_n + \frac{\Delta t}{2}\left[(-\alpha y_n) - \alpha(1 - \alpha\Delta t) y_n\right]$$

Now check, how much 'y' is changed in next time step from the previous one.

i.e. $\dfrac{y_{n+1}}{y_n}$ → define as Amplification factor G.

$$G := \frac{y^c_{n+1}}{y_n} = 1 - \alpha\frac{\Delta t}{2} - \alpha\frac{\Delta t}{2} + \alpha^2\frac{\Delta t^2}{2}$$

i.e. $G = 1 - \alpha\Delta t + \frac{1}{2}(\alpha\Delta t)^2$

For stable results you should have: $|G| \le 1$

For this: $\underline{\alpha \Delta t \le 2}$

Example

The following IV-ODE is given: $\dfrac{dT}{dt} = -\alpha(T^4 - T_a^4)$

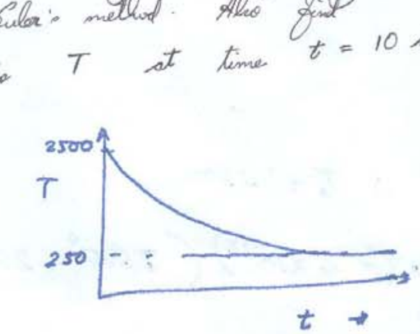
$T(0.0) = T_0 = 2500.0$

$T_a = 250.0$

$\alpha = 4.0 \times 10^{-12}\ (K^3 s)^{-1}$

Find the stability criteria for using modified Euler's method. Also find temperature T at time $t = 10$ seconds.

Soln.



→ The temperature decreases and reaches an asymptotic value of 250 K at infinite time.

→ Initial temp = 2500 K

i.e. Stability Criteria

$$T_{n+1}^{P} = T_n + \Delta t \, f_n$$

where $f_n = -\alpha\left(T_n^4 - 250^4\right)$

i.e. $= -(4.0 \times 10^{-12})\left[T_n^4 - 250^4\right]$

$$\therefore \quad T_{n+1}^{P} = T_n + \Delta t.\,(-4.0 \times 10^{-12})\left(T_n^4 - 250^4\right)$$

$$T_{n+1}^{P} = T_n + \Delta t\left(0.015625 - 4 \times 10^{-12}\, T_n^4\right)$$

i.e. $\cancel{T_{n+1}^{P} = T_n\left(1 + \Delta t\right.}$

$$T_{n+1}^{c} = T_n + \frac{\Delta t}{2}\left[-\alpha\left(T_n^4 - 250^4\right) + (-\alpha)\left(T_{n+1}^{P\,4} - 250^4\right)\right]$$

i.e. $\cancel{T_{n+1}^{c} = T_n - \frac{\alpha \Delta t}{2}\left[\left(T_n^4 - 250^4\right) + \left(T_n + \Delta t(-\alpha)(T_n^4 - 250^4)\right)^4\right]}$

$$T_{n+1}^{c} = T_n + -\frac{\alpha \Delta t}{2}\left[T_n^4 - 250^4 + \left(T_n + \Delta t\left(0.015625 - \alpha T_n^4\right)\right)^4 - 250^4\right]$$

From this you need to check.

$$\frac{T_{n+1}^{c}}{T_n} < 1.0$$

Put $\Delta t = 10.0$ s, $\therefore T_0 = 2500$

$$T_1^{P} = 2500 + 10.0 \times \left(-4 \times 10^{-12}\right)\left(2500^4 - 250^4\right)$$

$$= \underline{937.656} \;\; K$$

$$\therefore \; f^{P}_{n+1} = -4\times10^{-12}\left[937.656^4 - 250^4\right]$$

$$= -3.0763$$

$$\therefore \; T^{c}_{n+1} = T^{c}_{1} = 2500 + \frac{10}{2}\left[-156.23 + (-3.076)\right]$$

$$= \underline{1703.47 \;\; K}$$

Runge - Kutta Methods

As you have seen for IV-ODE

$$\frac{dy}{dt} = f(t, y)$$

You are providing finite-difference equation

$$y_{n+1} = y_n + \Delta t\, f_n \qquad \text{Explicit}$$

or

$$y_{n+1} = y_n + \Delta t\, f_{n+1} \qquad \text{Implicit}$$

i.e. $y_{n+1} = y_n + \Delta y$

Now this Δy can be given as:

$$\Delta y = C_1\,\Delta y_1 + C_2\,\Delta y_2 + \dots + \dots$$

a weighted sum of several Δy's.

$C_i \rightarrow$ Weighing factors.

Each Δy_i is evaluated as

$$\Delta y_i = \Delta t \underbrace{f(t, y)}$$

This $f(t,y)$ is evaluated at some point in the range $t_n \le t \le t_{n+1}$

For Example,

The second-order R-K Method is:

$$y_{n+1} = y_n + c_1 \Delta y_1 + c_2 \Delta y_2$$

where $\Delta y_1 = \Delta t \, f_n$ (Recall in Explicit Euler $y_{n+1} = y_n + \underbrace{\Delta t f_n}_{\Delta y_1}$)

$\Delta y_2 \rightarrow$ Based on $f(t, y)$ evaluated in the interval $t_n \le t \le t_{n+1}$

i.e $\Delta y_2 = \Delta t \, f\left(t_n + \alpha \Delta t, \; y_n + \beta \Delta y_1\right)$

One need to find the values of α and β the fractions ~~discussing~~ of increase in time and y.

$$\therefore \quad y_{n+1} = y_n + c_1 \Delta t f_n + c_2 \cdot \Delta t \, f\left(t_n + \alpha\Delta t, y_n + \beta\Delta y_1\right) \rightarrow (1)$$

~~Expan~~ Keeping the time grid point 'n' as base point

$$f(t, y) = f_n + \Delta t \left.\frac{\partial f}{\partial t}\right|_n + \frac{\Delta t^2}{2} \frac{d^2 f}{dt^2} + \Delta y \cdot \left.\frac{\partial f}{\partial y}\right|_n + \cdots$$

if $t = t_n + \alpha \Delta t$, ~~then~~ and $y = y_n + \beta \Delta y_n$

Then we have

$$f(t_n + \alpha\Delta t,\ y_n + \beta\Delta y_1) = f_n + \alpha\Delta t \left.\frac{\partial f}{\partial t}\right|_n + \beta\Delta y_1 \left.\frac{\partial f}{\partial y}\right|_n + O(\Delta t^2) \quad \hookrightarrow (2)$$

Substituting this expression in (1) we get

$$y_{n+1} = y_n + c_1\Delta t f_n + c_2\Delta t.\left[f_n + \alpha\Delta t \left.\frac{\partial f}{\partial t}\right|_n + \beta\Delta t f_n \left.\frac{\partial f}{\partial y}\right|_n + \cdots\right]$$

$$= y_n + (c_1 + c_2)\Delta t f_n + c_2\Delta t^2.\left(\alpha \left.\frac{\partial f}{\partial t}\right|_n + \beta f_n \left.\frac{\partial f}{\partial y}\right|_n + \cdots\right) \quad \hookrightarrow (3)$$

$\Rightarrow$ The unknown quantities are c_1, c_2, α, and β.

Using Taylor's series at base time is

$$y_{n+1} = y_n + \Delta t \left.\frac{dy}{dt}\right|_n + \frac{\Delta t^2}{2}\left.\frac{d^2y}{dt^2}\right|_n + \cdots\cdots$$

Now $\left.\frac{dy}{dt}\right|_n = f(t_n, y_n) = f_n$

$$\therefore \left.\frac{d^2y}{dt^2}\right|_n = \left.\frac{d}{dt}\left(\frac{dy}{dt}\right)\right|_n = \left.\frac{df}{dt}\right|_n = \left.\frac{\partial f}{\partial t}\right|_n + \left.\frac{\partial f}{\partial y}\right|_n \left.\frac{dy}{dt}\right|_n$$

$$= \left.\frac{\partial f}{\partial t}\right|_n + f_n \left.\frac{\partial f}{\partial y}\right|_n$$

$$\therefore y_{n+1} = y_n + \Delta t . f_n + \frac{\Delta t^2}{2}\left[\left.\frac{\partial f}{\partial t}\right|_n + f_n \left.\frac{\partial f}{\partial y}\right|_n\right] + \cdots \quad \hookrightarrow (4)$$

Comparing (3) and (4) we get:

$$c_1 + c_2 = 1$$

$$c_2\alpha = \frac{1}{2}$$

$$c_2\beta = \frac{1}{2}$$

So there are many possibilities of c_1, c_2, α, and β.

If we put $c_1 = \frac{1}{2}$

Then $c_2 = \frac{1}{2}$, $\alpha = 1$, and $\beta = 1$

Then $\Delta y_1 = \Delta t\, f_n$

$$\Delta y_2 = \Delta t .\, f(t_n + \Delta t,\ y_n + \Delta y_1)$$
$$= \Delta t\, f_{n+1}$$

$$\therefore y_{n+1} = y_n + c_1 \Delta y_1 + c_2 \Delta y_2$$
$$= y_n + \tfrac{1}{2} \Delta t\, f_n + \tfrac{1}{2} \Delta t\, f_{n+1}$$

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f_n + f_{n+1} \right]$$

$\Rightarrow$ This is same as Modified Euler method.

Fourth- Order R-K Method

One of the most famous R-K method.

The final form is:

$$y_{n+1} = y_n + \frac{1}{6}\left[\Delta y_1 + 2 \Delta y_2 + 2 \Delta y_3 + \Delta y_4 \right]$$

where
$$\Delta y_1 = \Delta t\, f_n$$
$$\Delta y_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta y_1}{2}\right)$$
$$\Delta y_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2},\ y_n + \frac{\Delta y_2}{2}\right)$$
$$\Delta y_4 = \Delta t\, f\left(t_n + \Delta t,\ y_n + \Delta y_3\right)$$

Example
For the previous example $\frac{dT}{dt} = -\alpha(T^4 - T_a^4)$

$T_0 = 2500\,K$, $\alpha = 4\times10^{-12}$, $T_a = 250\,K$

Using $\Delta t = 10$ seconds, find temperature at $t = 10$ second. By 4th order R-K method.

Soln.
We have $\Delta t = 10$ seconds, $T_0 = 2500$, $t_0 = 0.0$ s.

Now $t = 10$ second

$t_1 = t_0 + \Delta t = 10$ seconds

To evaluate T_1.

$$T_{n+1} = T_n + \frac{1}{6}\left[\Delta T_1 + 2\Delta T_2 + 2\Delta T_3 + \Delta T_4\right]$$

$$f = -\alpha(T^4 - 250^4)$$

$$\Delta T_1 = \Delta t\, f_0 = -10\times4\times10^{-12}\times(2500^4 - 250^4)$$
$$= -1562.34$$

$$\Delta T_2 = \Delta t.\, f\left(t_0 + \frac{\Delta t}{2}, T_0 + \frac{\Delta T_1}{2}\right) = 10.\, f(5, 1718.83)$$
$$= -10\times4\times10^{-12}\times\left(\cancel{937.656}^4\ 1718.83^4 - 250^4\right)$$
$$= -348.98$$

$$\Delta T_3 = \Delta t\, f\left(t_0 + \frac{\Delta t}{2}, T_0 + \frac{\Delta T_2}{2}\right) = 10.\, f(5, 2325.51)$$
$$= -10\times4\times10^{-12}\times(2325.51^4 - 250^4)$$
$$= -1169.70$$

$$\Delta T_4 = \Delta t.\, f(t_0 + \Delta t, T_0 + \Delta T_3)$$
$$= 10.\, f(5, 1330.30)$$
$$= 10.\,(-4\times10^{-12})\times(1330.30^4 - 250^4)$$
$$= -125.117$$

$$\therefore \; T_1 = T_0 + \frac{1}{6}\left[\Delta T_1 + 2\Delta T_2 + 2\Delta T_3 + \Delta T_4\right]$$

$$= 2500 + \frac{1}{6}\Big[-1562.34 + 2\times(-348.98)$$
$$+ 2\times(-1169.70) + (-125.117)\Big]$$

$$= 2500 - 787.47$$

$$= \underline{\underline{1712.53}} \; K$$