Lecture 33: Finite-Difference Method for IV-ODE

(17-Oct-2012)

LECTURE 33

17-10-2012

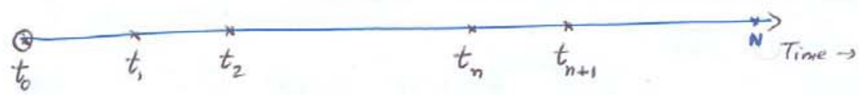
Finite-Difference Method for IV-ODE

* Yesterday we had seen the finite-difference method using first-order approximations for the IV-ODE

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

* When using first-order forward difference formula in the above IV-ODE, we got

Explicit Euler Method: $\cancel{y_{n+1}}$



$$y_{n+1} = y_n + \Delta t\, f_n \quad \rightarrow O(\Delta t^2)$$

If the end time is say t_N

Then $N = \dfrac{t_N - t_0}{\Delta t}$

$$\therefore \; y_N = y_{N-1} + \Delta t\, f_{N-1}$$

$$= y_{N-2} + \Delta t\, f_{N-2} + \Delta t\, f_{N-1}$$

$$= y_{N-3} + \Delta t\, f_{N-3} + \Delta t\, f_{N-2} + \Delta t\, f_{N-1}$$

$$\vdots$$

$$= y_0 + \sum_{n=0}^{N-1} \Delta t\, f_n \quad \Rightarrow \left\{ y_0 + \sum_{n=0}^{N-1} \Delta t\, f_n \right\}$$

The Implicit Euler Method

For the same IV-ODE $\frac{dy}{dt} = f(t, y)$; $y(t_0) = y_0$

$$\left.\frac{dy}{dt}\right|_{t_{n+1}} = f\left(t_{n+1}, y_{n+1}\right)$$

i.e. $$\frac{y_{n+1} - y_n}{\Delta t} = f_{n+1}$$

or $$\boxed{y_{n+1} = y_n + \Delta t\, f_{n+1}} \qquad O(\Delta t^2)$$

Comparing Explicit and Implicit Euler Methods

Why do we need implicit method?

or

Why do we go for explicit methods?

As seen from the finite-difference equations

$$y_{n+1} = y_n + \Delta t\, f_n \quad \rightarrow (1)$$

and $$y_{n+1} = y_n + \Delta t\, f_{n+1} \quad \rightarrow (2)$$

Equation (1) is straight-forward and you can get the unknown y_{n+1} using values of y_n.

In Equation 2 you need to solve for y_{n+1} as f_{n+1} is also an unknown quantity.

$\therefore$ Computations in implicit methods may be more.

Then why do we require implicit method.

Case

Consider the IV-ODE (homogeneous)

$$\frac{dy}{dt} + y = 0 \longrightarrow (3)$$

From you mathematics background

It is quite obvious the solution is $y = e^{-t}$

That is as the time increases, the value of y increases and the maximum value of $y = 1.0$.

$\Rightarrow$ Let us solve that IV-ODE using explicit and implicit Euler methods.

Explicit: $\rightarrow$ $y_{n+1} = y_n + \Delta t f_n$

As $f_n = -y_n$, we get

$$y_{n+1} = y_n + \Delta t\,(-y_n)$$

$$y_{n+1} = (1 - \Delta t)\,y_n$$

$\Rightarrow$ In this case your time step Δt should be $\Delta t < 1.0$ for realistic values.

$\Rightarrow$ For $\Delta t > 2.0$, the solution oscillates in two directions. We will not get stable solutions.

Implicit :- $y_{n+1} = y_n + \Delta t\, f_{n+1}$

Now $f_{n+1} = -y_{n+1}$ (for this problem).

$\therefore\ y_{n+1} = y_n + \Delta t\,(-y_{n+1})$

i.e. $(1 + \Delta t)\, y_{n+1} = y_n$

or $y_{n+1} = \dfrac{y_n}{1 + \Delta t}$

⇒ In this case, despite the value of $\Delta t > 1.0$, you may get some bounded solution. That is, the solution will not oscillate.

∴ Implicit method is unconditionally stable.

Requirement of Finite-Difference Methods

The FDM should be

→ Consistent
→ Stable
→ Convergent

A FD equation is consistent, if the difference between the finite-difference equation and the original ODE vanishes as $\Delta t \to 0$

Consider the IV-ODE

$$\frac{dy}{dt} + \alpha y = F(t) \quad ; \quad y(t_0) = y_0$$

If we use explicit Euler method:

$$y_{n+1} = y_n + \Delta t\, f_n$$

Now the IV-ODE at time t_n is:

$$\left.\frac{dy}{dt}\right|_{t_n} + \alpha y_n = F_n$$

$$\therefore f_n = F_n - \alpha y_n$$

and $$y_{n+1} = y_n + \Delta t\, F_n - \alpha \Delta t\, y_n \longrightarrow (1)$$

Consider y_n as base point and use Taylor's series

$$y_{n+1} = y_n + \Delta t \left.\frac{dy}{dt}\right|_{t_n} + \frac{1}{2!}\Delta t^2 \left.\frac{d^2y}{dt^2}\right|_{t_n} + \cdots\cdots \rightarrow (2)$$

From (1) and (2), we get

$$\Delta t \left.\frac{dy}{dt}\right|_{t_n} + \frac{\Delta t^2}{2}\left.\frac{d^2y}{dt^2}\right|_{t_n} + \cdots = \Delta t\, F_n - \alpha \Delta t\, y_n$$

Dividing by Δt:

$$\left.\frac{dy}{dt}\right|_{t_n} + \frac{\Delta t}{2}\left.\frac{d^2y}{dt^2}\right|_{t_n} + \cdots = F_n - \alpha y_n$$

Therefore, if we take the limit $\Delta t \to 0$

$$\left.\frac{dy}{dt}\right|_{t_n} + \alpha y_n = F_n$$ → The differential equation form at time t_n.

$\therefore$ Explicit Euler method is consistent. → This is same.

Convergence

The explicit Euler method $y_{n+1} = y_n + \Delta t\, f_n$

If $\frac{dy}{dt} + \alpha y = 0$

Then $f_n = -\alpha y_n$

$$\therefore \quad y_{n+1} = y_n + \Delta t(-\alpha y_n)$$

$$= (1 - \alpha \Delta t)\, y_n$$

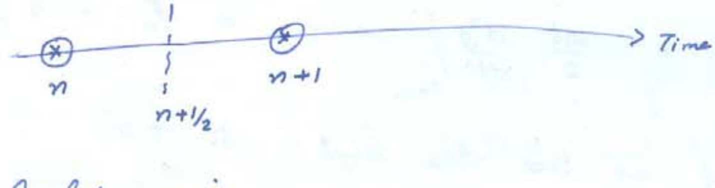
For stability $\quad -1 \leq (1 - \alpha \Delta t) \leq 1$

When this stability and consistence criteria is maintained, you will see the FDM converges.

Second - Order Single Point

Earlier we used the first order approximations for the derivatives in IV-ODE

Now keep the base-point as $n + 1/2$



Using Taylor's series

$$y_n = y_{n+1/2} + \left(\frac{-\Delta t}{2}\right) \frac{dy}{dt}\bigg|_{t_{n+1/2}} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \frac{d^2y}{dt^2}\bigg|_{t_{n+1/2}} + \cdots$$

$$y_{n+1} = y_{n+1/2} + \frac{\Delta t}{2} \frac{dy}{dt}\bigg|_{n+1/2} + \frac{1}{2} \frac{\Delta t^2}{4} \frac{d^2y}{dt^2}\bigg|_{n+1/2} + \cdots$$

$$\therefore \left.\frac{dy}{dt}\right|_{n+1/2} = \frac{y_{n+1} - y_n}{\Delta t} + O(\Delta t^2)$$

i.e $\left.\frac{dy}{dt}\right|_{n+1/2} + O(\Delta t^2) = f(t_{n+1/2}, y_{n+1/2}) = f_{n+1/2}$

$$\therefore y_{n+1} = y_n + \Delta t\, f_{n+1/2} \rightarrow O(\Delta t^3)$$

How will you obtain $f_{n+1/2}$

Now use:

$$\boxed{\begin{aligned} y^P_{n+1/2} &= y_n + \frac{\Delta t}{2} f_n \\ y^C_{n+1} &= y_n + \Delta t\, f^P_{n+1/2} \end{aligned}}$$

This is a two-step predictor - corrector method.
It is also called Modified Mid-point Method.

We have seen earlier

$$y_{n+1} = y_n + \Delta t\, f_{n+1/2} \quad O(\Delta t^3)$$

Use Taylor's series for f_{n+1} & f_n using $f_{n+1/2}$ as base.

$$f_{n+1} = f_{n+1/2} + \frac{\Delta t}{2} \left.\frac{df}{dt}\right|_{n+1/2} + O(\Delta t^2)$$

$$f_n = f_{n+1/2} - \frac{\Delta t}{2} \left.\frac{df}{dt}\right|_{n+1/2} + O(\Delta t^2)$$

$$\therefore f_{n+1/2} = \frac{1}{2}(f_n + f_{n+1}) \rightarrow O(\Delta t^2)$$

$$\therefore \boxed{y_{n+1} = y_n + \frac{\Delta t}{2}\left[f_n + f_{n+1} \right]} \rightarrow O(\Delta t^3)$$

This is implicit trapezoid finite difference equation.

$\Rightarrow$ The implicit scheme can be simplified as:

$$\boxed{\begin{aligned} y_{n+1}^{P} &= y_n + \Delta t\, f_n \\ y_{n+1}^{C} &= y_n + \frac{\Delta t}{2}\left[f_n + f_{n+1}^{P} \right] \end{aligned}}$$

$y_{n+1}^{P} \rightarrow$ Predictor value.

f_{n+1}^{P} evaluated using y_{n+1}^{P}

$\rightarrow$ This is modified Euler's finite-diff. equation.

Quiz

Show the modified Euler's method of finite-difference equations are consistent for the IV-ODE

$$\frac{dy}{dt} + \alpha y = F(t) \quad ; \quad y(t_0) = y_0$$

Please note that the quiz question asked in the class was to show implicit Euler method is consistent (and not the modified Euler's method).