LECTURE 32
16-10-2012

ORDINARY DIFFERENTIAL EQUATIONS

In the last few classes we had gone through various methods on

→ Numerical Differentiation
→ Numerical Integration

So where are these ~~two~~ two numerical techniques applied?

→ In some of the cases one ~~need~~ can to directly apply the techniques

→ e.g. To find area under a curve
→ To find slope of a function, etc.

* In most of the engineering and science problems the definition of the problem are mathematically represented by ordinary differential equations or partial differential equations.

* Therefore, one may require to solve these differential equations to understand the behaviour of the system or nature.

* Again to solve differential equations the differential equations consists of derivatives

Numerical Differentiation and Numerical Integration have to be employed for solving such differential equations.

I hope you are aware of the general features of differential equations:

* Order of a differential equation : $\frac{dy}{dx} = f(x, y)$

  → Order of highest derivative

* A general n^{th} order ODE

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = F(x)$$

* There are
  → Initial - Value ODE's
  → Boundary - Value ODE's

* Linear ODE :→ All derivatives appear in linear form.

  e.g. $\frac{dy}{dx} + \alpha y = F(x)$

* Homogeneous and Non-homogeneous differential equations

  e.g.: $\frac{dy}{dx} + \alpha y = 0$ Linear, first-order homogeneous ODE.

  $\frac{dy}{dx} + \alpha y = F(t)$ linear, first-order, non-homogeneous ODE

  ↳ Forcing term

* There can be system of differential equations

A general non-linear first-order ODE →

$$\frac{dy}{dx} = f(x,y)$$

A general non-linear second-order ODE →

$$\frac{d^2y}{dx^2} + P(x,y)\frac{dy}{dx} + Q(x,y)\,y = F(x)$$

↳ If all auxiliary conditions specified at the same value of independent variable → then you need to march forward. This is Initial Value

↳ If auxiliary conditions are specified at end points of independent variable, etc., then Boundary Value Problems.

The Physical Problems

→ Propagation problem (like IV-problem)

→ Equilibrium problems

→ Eigen Problems

An IV ODE → $\frac{dy}{dt} = f(t,y)$; $y(t_0) = y_0$

A BV ODE →

$$\frac{d^2y}{dx^2} + P(x,y)\frac{dy}{dx} + Q(x,y)\,y = F(x)$$

$$y(x_1) = y_1$$
$$y(x_2) = y_2$$

One-Dimensional Initial Value ODE's

Consider the heat transfer mechanism. You may describe the heat transfer problem using IV-ODE

$$\frac{dT}{dt} = -\alpha\left(T^4 - T_a^4\right) \quad ; \quad T(0) = T_0$$

→ This problem can be solved by our numerical techniques.

Using finite-difference method (FDM)

→ The objective of FDM :

→ Discretize the continuous domain to discrete finite-different grid.

→ Approximate exact derivatives by algebraic finite-difference formulas

→ Substitute them to the original ODE and subsequently obtain algebraic ~~D~~ equation.

→ Solve the resulting algebraic equation.

→ There are three major types of finite-difference method for IV-ODE.

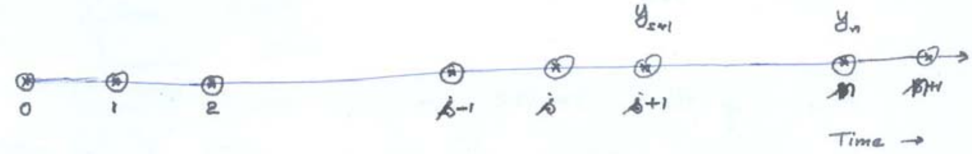
* Single-point methods
* Extrapolation methods
* Multi-point methods

→ The data from one grid point is used in evaluating the next grid point → Single Point

→ Data from more than one previous grid points used in evaluating a grid point → Multi-point method.

Finite- Difference Method

As suggested earlier the continuous physical (or temporal) domain is discretised into finite-different grids.



The figure above shows finitely-gridded temporal domain.

→ The grid points can be equally spaced and/or unequally spaced.

→ Let us start with uniform grid size Δt

$$\left.\frac{dy}{dt}\right|_{t_n} = y'_n \quad \text{(Just Representation)}.$$

Using the data at y_n, one can ~~obtain~~ through Taylor's series define y_{n+1}

→ We have studied numerical differentiation

∴ First order forward-difference formula for

$$\left.\frac{dy}{dt}\right|_{t_n} = \frac{y_{n+1} - y_n}{\Delta t} + O(\Delta t)$$

Similarly a first-order backward-difference formula for

$$\left.\frac{dy}{dt}\right|_{t_{n+1}} = \frac{y_{n+1} - y_n}{\Delta t} + O(\Delta t)$$

We can also define a second-order centered difference formula for $\frac{dy}{dt}$ at the time $n+\frac{1}{2}$

$$\left.\frac{dy}{dt}\right|_{t_{n+1/2}} = \frac{y_{n+1} - y_n}{\Delta t} + O(\Delta t^2)$$

Now if the IV ODE is

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(0) = y_0$$

Then $\left.\frac{dy}{dt}\right|_{t_n} = \frac{y_{n+1} - y_n}{\Delta t}$ or $\left.\frac{dy}{dt}\right|_{t_{n+1}} = \frac{y_{n+1} - y_n}{\Delta t}$

$$\therefore \frac{y_{n+1} - y_n}{\Delta t} = f(t_n, y_n)$$

or

$$\underline{\underline{y_{n+1} = y_n + \Delta t\, f(t_n, y_n)}}$$

This is finite-difference algebraic equation.

Smoothness

If the problem (that is to be solved) has discontinuous derivatives, your finite-difference method may misbehave.

→ Problems not having discontinuities → Smoothly varying problems.

Errors

Using finite-difference methods, you may encounter following types of errors:

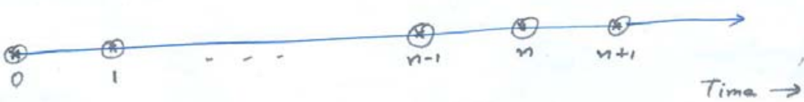
(i) Errors in initial data
(ii) Algebraic errors
(iii) Truncation errors
(iv) Round-off errors
(v) Inherited errors (or cumulated errors)

First - Order Approximations (Euler Method)

Consider the differential equation

$$\frac{dy}{dt} = f(t, y) \quad ; \quad y(t_0) = y_0$$

Discretise the time domain



Using first order forward difference

$$\left.\frac{dy}{dt}\right|_{t_n} = f(t_n, y_n)$$

i.e. $\dfrac{y_{n+1} - y_n}{\Delta t} = f_n \Rightarrow$ i.e $\dfrac{y_{n+1} - y_n}{\Delta t} + O(\Delta t) = f_n$

or $\boxed{y_{n+1} = y_n + \Delta t \, f_n} + O(\Delta t^2)$

This is Explicit Euler Method.

Suppose if the total time domain is from t_0 to t_N

$$\therefore \quad N = \frac{t_N - t_0}{\Delta t}$$

$$y_N = y_{N-1} + \Delta t\, f_{N-1}$$

$$= y_{N-2} + \Delta t\, f_{N-2} + \Delta t\, f_{N-1} = y_{N-2} + \Delta t \sum_{n=N-2}^{N-1} (\;)$$

$$= y_{N-3} + \Delta t\, f_{N-3} + \Delta t\, f_{N-2} + \Delta t\, f_{N-1}$$

$$\vdots$$

$$= y_0 + \sum_{n=1}^{N} \Delta t\, f_{N-n}$$

The total truncation error $= \sum_{n=1}^{N} \frac{1}{2} y''(\tau_n) \Delta t^2$

$$\rightarrow N \cdot \frac{1}{2} y''(\tau) \Delta t^2$$

$$\rightarrow \underline{O(\Delta t)}$$

Implicit Euler Method

$$\left.\frac{dy}{dt}\right|_{n+1} = \frac{y_{n+1} - y_n}{\Delta t} + O(\Delta t)$$

$$\Rightarrow y_{n+1} = y_n + \Delta t\, f_{n+1} \quad \rightarrow O(\Delta t^2)$$

As f_{n+1} depends on y_{n+1}, which is unknown, the scheme is implicit Euler method.